Regularisation of Chiral Gauge Theories By N.D. Hari Dass Institute of Mathematical Sciences, C.I.T Campus, Chennai 600 113

Abstract

This article gives a review of the topic of regularising chiral gauge theories and is aimed at a general audience. It begins by clarifying the meaning of chirality and goes on to discussing chiral projections in field theory, parity violation and the distinction between vector and chiral field theories. It then discusses the standard model of electroweak interactions from the perspective of chirality. It also reviews at length the phenomenon of anomalies in quantum field theories including the intuitive understanding of anomalies based on the Dirac sea picture as given by Nielsen and Ninomiya. It then raises the issue of a non-perturbative and constructive definition of the standard model as well as the importance of such formulations. The second Nielsen-Ninomiya theorem about the impossibility of regularising chiral gauge theories under some general assumptions is also discussed. After a brief review of lattice regularisation of field theories, it discusses the issue of fermions on the lattice with special emphasis on the problem of species doubling. The implications of these problems to introducing chiral fermions on the lattice as well as the interpretations of anomalies within the lattice formulations and the lattice Dirac sea picture are then discussed. Finally the difficulties of formulating the standard model on the lattice are illustrated through detailed discussions of the Wilson-Yukawa method, the domain wall fermions method and the recently popular Ginsparg-Wilson method.

1 Introduction

'Chirality' simply means handedness. Handedness in its most basic meaning means a correlation between circular motion and linear motion. For example,when the head of a corkscrew is given a rotation, the tip of the screw moves forward or backward depending on which way the head is rotated. A quantity representing this correlation is the scalar product of angular momentum \vec{J} , the generator of rotations and momentum \vec{P} , the generator of translations. Actually the magnitudes of the angular momentum and linear momentum are irrelevant for quantifying the desired correlation. This quantity $\vec{J} \cdot \vec{P}$ is called 'helicity' of the particle.

For massive particles, it can easily be seen that helicity depends on the initial frame of reference. If the inertial frame is the rest frame of the massive particle, helicity is not even defined. The direction of the linear momentum can change depending on the frame while the direction of angular momentum does not. Hence helicity can take both signs.

But for massless particles helicity has a Lorentz-invariant meaning. No rest frame is available to these particles and consequently the direction of the momentum can not be reversed by a change of frames. This fact is of special significance in the case of fermions. Specifically, in d = 4, the Dirac-Weyl equation describing massless fermions is invariant under the ' γ_5 -transformation:

$$\delta\psi = i\epsilon\gamma_5\psi\tag{1}$$

As a consequence, γ_5 is conserved and its eigenvalues are the particle helicities. From now onwards we shall call the γ_5 -eigenvalues 'chirality'.

It also follows that we can introduce the chirality projection operators $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ and write

$$\psi_L = P_+ \psi$$

$$\psi_R = P_- \psi$$
(2)

2 Parity Violation

A development of great significance in physics was the discovery of parity violation. It was found that in β -decay, the electron was emitted in a direction predominantly anti-parallel to the direction of nuclear spin. In a landmark development Sudarshan, Marshak, Feynman, Gell-mann and Sakurai showed that the weak interaction currents were of the V - A type. We will see that this has had profound implications for the conceptual developments in particle physics. In its impact and timelessness this discovery should rank along with Galileo's inertia, Einstein's equivalence principle etc.

Mathematically speaking the weak interaction current has the form

$$J^+_{\mu} = \bar{p}_L \gamma_{\mu} n_L \tag{3}$$

The remarkable feature of this current is that it is made of only the lefthanded fields. In fact it is a property of γ_{μ} -interactions that they preserve chirality. It is instructive to compare the structure of the electromagnetic current also expressed in terms of the chiral components:

$$J^{el}_{\mu} = \bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R \tag{4}$$

Again the current does not mix the L and R components. However, in the electromagnetic current the L and R fields occur on an equal footing, reflecting the parity conserving nature of the electromagnetic interactions. Such theories will henceforth be referred to as **Vector** theories in contrast to the theory of weak interactions which shall be called **Chiral**.

As far as the structure of these currents are concerned, it appears possible to treat the L and R fields as independent species of particles. But as mentioned before, chirality does not have a Lorentz-invariant meaning for massive particles. In fact the mass term in a Lagrangean expressed interms of L and R fields looks like

$$m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \tag{5}$$

Thus inertia can be viewed as merely an interaction that switches L and R species! This is indeed a major paradigm shift in elementary particle theory.

3 The Standard Model of Weak and Electromagnetic Interactions

The next major development of relevance is the standard model which unified in one step the old Fermi theory of weak interactions, the V - A structure of weak currents inspired by the observed parity violation in beta decay, and remarkably, the electromagnetic interactions. Other highlights of this theory were the 'dynamical' origin of masses, existence of weak neutral currents and from a theoretical point of view, renormalisability.

The construction of the standard model is based on the symmetry group $SU(2)_L \times U(1)$. Restricting ourselves to the leptonic sector(electron, neutrino) for convenience, the basic fields are taken to be (e_L, ν_{e_L}) transforming as a doublet under $SU(2)_L$, and e_R transforming as a singlet under $SU(2)_L$. Both the left-handed doublet and the right-handed singlet transform non-trivially

under U(1).Furthermore, the U(1) charges of the L and R fields are different.The vector currents $\bar{\psi}_L \gamma_\mu \tau \psi_L$ transforming as a triplet under $SU(2)_L$ and $\bar{\psi}_L \gamma_\mu \psi_L, \bar{\psi}_R \gamma_\mu \psi_R$ couple to the gauge fields \vec{W}_μ and B_μ . The neutral currents couple to the U(1)-gauge field in proportion to the hypercharges of the L and R fields.

It should be emphasised here that the parity violations are "put in by hand" in the standard model. This is realised by ascribing very different properties (like hypercharges, $SU(2)_L$ representations) to the R and L fields. These difference do not arise as manifestations of any deeper dynamics. Whenever l and R fields transform differently under the group, one says that there are complex representations of the group.

Associated with the gauge fields are the following **local** gauge transformations under which the theory is invariant. In addition to the fermionic fields and gauge fields, the standard model also has the so called Higgs field which transforms as a complex $SU(2)_L$ doublet. The four real components of this complex doublet can also be arranged as a 2×2 matrix Φ :

Likewise, the gauge fields \vec{W}_{μ} can be equivalently represented as a matrix $\mathbf{W}_{\mu} = \vec{\mathbf{W}}_{\mu} \cdot \vec{\tau}$. In terms of these fields and the fermionic fields the gauge transformations are:

$$B'_{\mu}(x) = B_{\mu}(x) + \partial_{\mu}\theta(x)$$

$$\vec{\mathbf{W}}'_{\mu}(x) = g(x)\mathbf{W}_{\mu}g^{-1}(x) - i\partial_{\mu}g(x) \cdot g^{-1}(x)$$

$$\psi'_{L}(x) = g(x)e^{iy_{L}\theta(x)}\psi_{L}(x)$$

$$\psi'_{R}(x) = e^{iy_{R}\theta(x)}\psi_{L}(x)$$

$$\Phi'(x) = e^{i(y_{L}-y_{R})\theta(x)}g(x)\Phi$$
(7)

These transformation rules allow for gauge-invariant interactions of the type

$$\mathcal{L}_{Yukawa} = g_Y \bar{\psi}_L \Phi \psi_R + h.c \tag{8}$$

It is easily recognised that the same interaction can generate masses for the fermions(see eq()) if the Higgs field Φ develops a vacuum expectation value.But that would mean that the **global** part of $SU(2)_L \times U(1)$ would be spontaneously broken and by the Goldstone theorem there ought to be massless Goldstone bosons equalling **at least** the number of broken generators. The latter is estimated on noting that $SU(2)_L$ -rotations about the direction along which the VEV of Φ points still leave the action invariant. Thus there are three broken generators and in perturbative analysis one finds three Goldstone bosons. Because the global invariance is elevated to a local one, the relevant phenomenon is the Anderson-Higgs mechanism by which three of the vector bosons acquire masses and the gauge boson corresponding to the unbroken generator is identified as the massless photon. This is inded the conceptual economy of the standard model which , while unifying weak and electromagnetic interactions, naturally realises their importance, namely, the difference in the ranges of the interactions.

As already stressed, the standard model is perturbatively **renormalis-able**. This hinges on the delicate fact that breaking the symmetries spontaneously does not spoil renormalisability and that the symmetric version of the theory is renormalisable. In fact one of the biggest stumbling blocks towards the construction of a field theory of weak interactions was the non-renormalisability of **generic** massive Yang-mills theories. Establishing this consists of first regularising the theory , then checking all the Ward identities for the regularised theory and finally using definitions of an optimal set of observables to "renormalise". One has the choice of either regularising the theory maintaining all the symmetries (if possible), in which case the Ward identities are automatically satisfied in the regularised theory, or, using non-invariant regularisation schemes and adding the requisite (non-invariant)counter terms to realise the Ward identities.

One of the popular techniques for regularisation is the so-called dimensional regularisation where the dimension of space-time is taken to be $n = 4 - \epsilon$ and removing cut-off is equivalent to taking the limit $\epsilon \to 0$. This way of regularising has the advantage that it is manifestly gauge invariant.

But already at this stage chirality begins to pose some problems. The origin of this difficulty lies in the fact that chirality is a very dimension dependent concept i.e in odd dimensions there are no Weyl fermions. More explicitly, γ_5 which enters the chiral projection operators (in d = 4) is given by $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ and it is clear that there is no straightforward way of generalising this to arbitrary dimensions. However, there is a procedure due to Breitenlohner and Maison, which seems to succesfully address this question , at least in low orders of perturbation theory.

3.1 A Caveat : The Anomalies

The renormalisability of the standard model hinges on a caveat of the absence are cancellation of the so called 'anomalies'.Conceptually,anomalies will play a vital role in the rest of our discussions, so it is worth our while to examine them carefully.Anomaly is an effect in Quantum Field Theory whereby a symmetry is destroyed by the quantum fluctuations.It is highly counterintuitive as it is not clear why quantum fluctuations should invalidate conservation laws.

On closer examination, anomalies can be understood as arising because of infinite number of degrees of freedom characteristic of all field theories. It should be recalled that the occurrence of divergences in QFT necessitating the renormalisation procedure is also a consequence of these infinitely many degrees of freedom. In the case of a theory like Quantum Electrodynamics or Quantum Yang-Mills theory, regularisation and renormalisation can be carried out preserving the corresponding gauge invariances. If, however, we consider a theory with several invariances at the so-called 'classical' level, the likelihood of there being no way of regularising the theory maintaining all the ward identities can not be ruled out. Then a choice has to be made to give up some of the invariances. This in a nutshell is the basis of the anomalies.

Before we analyse the anomalies in depth, it is relevant to point out that not all invariances need be symmetries in the Wignerian sense, which, among other things, would associate degeneracies with symmetries. Therefore, contrary to the oft-used language, local gauge invariances are not symmetries and hence the expression 'gauge symmetry' is an abuse of language. What local gauge invariances represent are statements about the number of degrees freedom or more precisely, they specify the physical configuration space of the theory. A rigid or global gauge invariance, on the other hand, is a symmetry of the theory. Instead of leading to degeneracies, it leads to superselection rules.

Let us now take a more detailed look at the anomalies.For that purpose,let us consider a theory which is classically(in the sense of ignoring the effect of quantum fluctuations) is invariant under

$$\delta \psi(x) = \alpha(x)\psi(x)$$

$$\delta A_{\mu} = \partial_{\mu}\alpha(x)$$

$$\delta \psi(x) = \beta \psi(x)$$
(9)

with the associated conservation laws

$$\partial_{\mu}j^{\mu} = 0 \partial_{\mu}j^{5}_{\mu} = 0$$
 (10)

The big surprise was that no regularisation scheme could be found that maintained the ward identities corresponding to both the gauge transformations tions ($\alpha(x)$ - transformations) and the chiral transformations(β -transformations). Consequently if $\partial_{\mu}j^{\mu}_{ren} = 0$, then $\partial_{\mu}j^{\mu}_{5,ren} \neq 0$.

In the example considered above, chiral symmetry was global. In the standard model, the invariances are local. Also, for simplicity we only considered Abelian transformations. But in the standard model we have both Abelian and non-Abelian transformations. It turns out that when the non-Abelian group is SU(2), there can not be any non-conservation of the non-Abelian currents (in d = 4) and one should only worry about the U(1) anomaly.

It is clear that when the current coupling to a gauge field is anomalous, gauge invariance is lost. As stated earlier, gauge invariance can be viewed as a statement about the degrees of freedom of a theory. Thus if the object is to construct a consistent theory with the required number of degrees of freedom, anomalies would render such a theory sick.

Indeed the general folklore was that anomalous gauge theories are sick and ill-defined. If ,however, we take the view point that the true degrees of freedom could be larger than the naive count of degrees of freedom, anomalous gauge theories could in principle be consistent.

That this could indeed be the case was shown by Jackiw and Rajaraman. They showed that in d = 2 anomalous gauge theories can indeed be consistent, and the price for the consistency were additional degrees of freedom. Whether or not the same thing works in higher dimensions is still an open issue, though the chances seem remote.

Now the caveat in proofs of renormalisability is that the presence of anomalies leads to additional sources of divergences which can not be absorbed into the allowed set of counterterms. This was first pointed out by Gross and Jackiw.

The miracle of the standard model is that the full theory including quarks and leptons is actually anomaly free and the above-mentioned caveat is no longer of any concern.

3.2 A Non-perturbative Definition of the Standard Model?

So far our analysis of the standard model, a chiral gauge theory, has been perturbative. But it is well known that perturbative analyses can be highly misleading as for example in the $\lambda \phi^4$ theory in d = 4 where perturbation theory yields a non-trivial S-matrix but a fully non-perturbative analysis shows the theory to be trivial (technically the proofs are still a bit incomplete in d = 4). Of course there are indications of this theory being problematic through the appearance of so-called Landau ghosts in perturbative analalysis. But it is not clear whether these are artefacts of perturbation theory.

In the case of Quantum Electrodynamics, predictions of perturbation theory are very well borne out experimentally. That theory too suffers from the presence of Landau ghosts, so one can not immediately conclude from the presence of Landau singularities that perturbative analysis is unreliable.

In the case of the standard model also, predictions of perturbation theory are in excellent agreement with observations. So perturbative analysis is perhaps not as misleading as in the case of $d = 4\lambda\phi^4$ theory.

Nevertheless a non-perturbative formulation of the theory that is mathematically well defined is always desirable.From the point of view of confronting the theory with experiments also such a formulation is desirable as it will make hitherto inaccessible aspects of the theory amenable for verification.From a matter of principle also one should insist on such a formulation because for the theory to make sense it should be well defined in all regions of its parameter space i.e both perturbative and non-perturbative regions.

So we pose the following two questions about the standard model:

1. Is there a nonperturbative definition of the theory?

2. If so, can that definition be a constructive one?

Before answering these questions it is instructive to take another look at the anomalies. We shall closely follow the reasonings of Nielsen and Ninomiya[1],who wanted to arrive at an 'intuitive understanding' of the anomalies.In particular, to clarify the mystery of "how a classical conservation law disappears quantum mechanically?".

Following them let us start with Weyl-particles in 1 + 1 dimensions. The dispersion relations are given by

$$\omega = \pm p \tag{11}$$

where the \pm refere respectively to right and left movers.Let these Weyl particles carry charges and consider the action of an electric field F_{01} .Without loss of generality let $qF_{01} > 0$ where q is the charge of the Weyl particles.

3.3 Single Particle Picture

If one restricted attention to the single particle sector with positive energy, t is easy to see that under the influence of the electric field, the particle momentum will steadily increase with time. For the right-movers this will imply a steadily increasing energy while for the left-movers the energy steadily decreases. The motion is along the dispersion curve, as shown in the figures below. This is also called a "spectral flow".

Clearly no net chirality is produced; by net chirality we mean the differ-



Figure 1: Single Particle Picture

ence in the number of right-movers and left-movers. It is also obvious that this picture remains if we consider a collection of left and right-movers where each individual particle is of positive energy. This is essentially the "classical" picture.

It is also clear that the above picture holds irrespective of whether the dynamics conserves parity or not as long as it conserves chirality. More precisely, instead of the parity conserving electromagnetic interaction

$$\mathcal{L}_{em} = (\bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R A_\mu \tag{12}$$

one had considered the parity violating interaction

$$\mathcal{L} = (g_1 \bar{\psi}_L \gamma_\mu \psi_L + g_2 \bar{\psi}_R \gamma_\mu \psi_R B_\mu \tag{13}$$

net chirality would still be preserved even though the rates at which the left and right-movers move along the spectral curve would be different. If on the other hand, the dynamics was itself chirality non-conserving even if parityconserving as in

$$\mathcal{L} = \bar{\psi}_L \Phi \psi_R + h.c \tag{14}$$

there would be net chirality production in proportion to the original chirality.

3.4 Influence of The Dirac Sea

However, we know that the single particle picture is quantum mechanically incomplete both for fermions and bosons. In the former case, one possible resolution is to invoke the concept of the "Dirac Sea" whereby all the physically undesirable negative energy states are completely filled. Pauli exclusion principle would then forbid transitions from the positive energy states into negative energy states thereby stabilising the positive energy states. However, it would always be possible to lift a particle in the Dirac sea to a positive energy state leaving behind a "hole" in the Dirac sea which would have positive energy relative to the Dirac sea state(vacuum state) and be oppositely charged compared to the electron. This is the "positron" state, popularly called the anti-particle state.

Though modern formulations of field theory exist which do not explicitly invoke the concept of the Dirac Sea, it is nevertheless instructive to analyse field theoretic phenomena in terms of the visually more transparent Dirac Sea.

Following Nielsen and Ninomiya, let us now take a fresh look at the effect of the electromagnetic fields on charged Weyl particles. The vacuum or the Dirac Sea has all the Right-handed particle states with negative momentum and all the Left-handed particle states with positive momentum fully occupied, as shown in the figure below: Using the spectral flow picture it is easy to see that when the electric field is applied on the vacuum state, there will be a steady creation of R-particles and at the same time a steady depetion of L-particles. Thus there will be net creation of chirality and the axial current j^5_{μ} is no longer conserved! But since the number of "holes" created is the same as the number of "particles" created, total electric charge is indeed conserved.

Thus we get a qualitative understanding of axial anomalies through the



Figure 2: The Dirac Sea Picture

Dirac Sea picture. Actually even quantitatively the correct anomaly follows. If the electric field has constant value E over a region of length l, one has

$$\frac{dn_L}{dt} = l\frac{qE}{h}$$

$$\frac{dn_R}{dt} = -l\frac{qE}{h}$$
(15)

where h is the Planck's constant. These equations immediately imply

$$\partial_{\mu}j^{5}_{\mu} = \frac{q}{\pi}E \tag{16}$$

which is the correct form of the anomaly.

3.5 The Four Dimensional Case

The extension of the above mentioned arguments to higher even-dimensional cases is straight forward.Let us consider the d = 4 case as an example.The Weyl equation now reads:

$$i\gamma^{\mu}D_{\mu}\psi = 0 \qquad D_{\mu} = \partial_{\mu} - iqA_{\mu} \qquad (17)$$

Iterating this equation twice one gets

$$\{-D_{\mu}D^{\mu} + \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]F_{\mu\nu}\}\psi = 0$$
(18)

Let us consider those field configurations for which $F_{\mu\nu}F^{\mu\nu} \neq 0$. By choosing an appropriate Lorentz-frame, it is possible to make $F_{01} \neq 0$ and $F_{23} \neq 0$ with all other components of $F_{\mu\nu}$ vanishing.On noting that $[\gamma_0, \gamma_1]$, $[\gamma_2, \gamma_3]$ and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ form a mutually commuting set, the spectrum can be labelled by their simultaneous eigenvalues. In the magnetic field F_{23} , the charged particles form Landau levels with degeneracy $\pi \frac{2F_{23}(Area)_{23}}{(2\pi)^2}$. Now because of the special choice of directions of the electric and magnetic fields, the 3 + 1-dimensional problem can be treated as if it were a 1 + 1-dimensional problem with the additional degeneracy. Thus

$$\frac{d(n_R - n_l)}{dt} = \frac{Vol}{(2\pi)^2} q F_{01} F_{23}$$
(19)

leading to

$$\partial_{\mu} j^{5}_{\mu} = \frac{q}{(2\pi)^{2}} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{20}$$

which is again the correct formula for the U(1)-anomaly.

It would be interesting to work out the non-abelian versions of these results.

Lesson: Anomaly is the continuing pumping out of the (infinite) Dirac Sea!Stated differently, it is the bottomlessness of the Dirac sea that allows pumping of net chirality without any paying any price.

3.6 Regularising the Dirac Sea

An infinitely deep Dirac Sea is clearly an unphysical idea arising out of the idealisation that energy and momenta can take arbitrary values. Another way of stating the crux of this matter is that the infinite Dirac Sea is tantamount to infinitely many degrees of freedom. It should be recalled that the ultraviolet divergences arising in Quantum Field Theory, necessitating the renormalisation procedure, always have their source in the assumption of infinitely many degrees of freedom. In fact, mathematically the theory is meaningless under these circumstances. Therefore one first works with a regularised version of a quantum field theory is a suitably and consistently truncated version of that theory.

Clearly, the Dirac Sea has also to be regularised or in other words provided with a "bottom". Alternatively, regularisation in QFT throws away the processes happening at the bottom of the sea as events occurring at a very high energy scale and not relevant to the physics at low energy scales. In particular, the contribution to the net chirality by the inflow at the bottom of the sea is ignored in the process of regularisation. Since the anomaly was



Figure 3: Truncated Dirac Sea

seen to be heavily dependent on the Dirac sea being infinite, one may wonder as to the fate of the anomalies if one were to regulate the Dirac Sea.

Indeed it is clear that when one regularises the Dirac sea, one *unavoidably* introduces dynamics which breaks the conservation of chirality **unless one invents a very special type of regularisation**. We shall see later that Lattice regularisation is special in this sense. Of course, care should be exercised as to how exactly the Dirac Sea is regularised as otherwise even electric charge and perhaps momentum also may not be conserved.

3.7 What Constitutes a Regularisation?

Essentially a regularisation is replacing the original continuum (but mathematically ill-defined) theory by one that is a very good approximation to it at large distance scales but is finite(and hence mathematically well-defined).

It is a very reasonable premise that the truly fundamental theory is mathematically well-defined and therefore **finite**. From a physical stand point also, it is reasonable to expect it to be finite as infinity is an idealisation never to be realised under actual physical circumstances. It is important to emphasise that by this reckoning the truly fundamental theory should be finite and not just **renormalisable** as in the latter case one hides some ignorance through the renormalisation procedure and can not qualify to be a truly fundamental theory. Thus the finite fundamental theory can be thought of as a regularisation for its lower energy effective theory.

Then the paradox of regularisation, namely, the unavoidable breaking of chirality conservation in a regularised theory (generic), will also be a paradox for any truly fundamental theory.

4 The Second Nielsen-Ninomiya Theorem

[2] Nielsen and Ninomiya proved two important theorems in the context of chirality. In the first they showed that it was impossible to put neutrinoes on a lattice without explicitly breaking chiral invariance. In the equally fundamental second theorem they showed that under rather general conditions it would be impossible to regularise **Chiral Gauge Theories**. In our discussion till now the gauge aspects have only been implicit to the extent that a classically conserved chiral charge was assumed but its coupling to a gauge field was not considered. The second NN-theorem deals with the difficulties of regularisation of chiral gauge theories.

Statement: It is not possible to simultaneously fulfill all of the following:

1. Fundamental Regularisation

The fundamental theory is taken to be a finite theory.

2. Parity Violation

Different number of right and left-handed species for given representations of symmetry groups.

3. Exact Gauge Invariance

This eans gauge invariance should be valid also at the regularisation scale. Veltman has argued that even "small" violations of gauge invariance at short distances can have large effects on S-matrix elements. This is also in conformity with our earlier remarks on gauge invariance being a specification of the degrees of freedom and that there can not be any meaning to the breaking of this invariance.

4. Bilinearity A technical assumption is made about the action being bilinear in the Weyl-fields.

4.1 Are Superstrings an Exception?

As commented by Nielsen and Ninomiya, Superstring theories seem to offer a way out of the no-go theorem. Superstring theories are finite, at least perturbatively. It is worth commenting on the sense in which these theories are finite. In these theories only the spectrum and S-matrix are calculable. The S-matrix is found to be finite in every order of perturbation theory without recourse to any renormalisation. However, the spectrum contains an **infinite** species of particles and momenta can take arbitrarily large values. It would be interesting to investigate whether non-perturbatively there are indeed **finite** degrees of freedom as befitting a truly fundamental theory.

Now the point is that some superstring theories have in their spectrum chiral fermions in complex representations as well as gauge field coupled to them.

To this extent superstring theories seem to evade the no-go theorem. How exactly do they achieve this? Is it that the no-go theorem is valid only in local field theories? Though superstring theories are not local field theories, they do have some locality properties in that the string-interactions are local. It will be interesting to fully understand this issue.

It should be emphasised that the possibility that some of these superstring theories may not turn out to be phenomenologically succesful is of no consequence to this discussion of matters of principle.

5 Lattice Regularisation

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Figure 4: An One Dimensional Regular Lattice

In the lattice regularisation space-time is approximated by a discrete set

of points i.e $x^{\mu} \to n^{\mu}a$ where n^{μ} is an integer-valued four-vector and a is the lattice spacing. A scalar field $\phi(x)$, for example, is represented by $\phi_{\mathbf{n}}$ where \mathbf{n} stands for the four-vector. It turns out to be useful to work with only dimensionless objects viz. $\phi_L = a\phi$ etc. The derivatives of fields are replaced by finite differences. For example,

$$\partial_{\mu}\phi(x) \to \frac{\phi_{\mathbf{n}'} - \phi_{\mathbf{n}}}{ma}$$
 (21)

where $\mathbf{n}' - \mathbf{n} = m\mathbf{e}^{\mu}$ and \mathbf{e}^{μ} is the unit vector in the μ -direction.

It is quite clear that the choice of lattice i.e hypercubic, triangular etc as well as the choice of finite difference chosen to approximate field derivatives are arbitrary. It is believed that in the continuum limit these differences should become irrelevant.

Some of the remarkable features of the lattice formulation are that it affords a manifestly gauge-invariant regularisation even for non-Abelian gauge theories. Furthermore, it produces a regularised theory while most regularisation schemes used in continuum quantum field theories regularise process by process as for example in Pauli-Villars regularisation of Feynman diagrams. This also means that lattice regularisation is a **non-perturbative** regularisation. Therefore it gives a non-perturbative formulation of the theory.

An important feature of Lattice regularisations is that momentum space is compact and is topologically a d-torus if the quantum field theory is formulated in d-spacetime dimensions. This will be seen to have a profound impact on regularising chiral gauge theories. More precisely, the Euler characteristic of the momentum space(Brilloin zone) is 0.

5.1 Fermions on the Lattice

[4] Consider the continuum Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi + m\psi = 0 \tag{22}$$

A possible candidate for the lattice equivalent of this is

$$i\gamma^{\mu}\Delta_{\mu}\psi + m\psi = 0 \tag{23}$$

where the (forward)shift operator Δ_{μ} is defined by

$$\Delta_{\mu} f(\mathbf{x}) = f(\mathbf{x} + a\mathbf{e}_{\mu}) - f(\mathbf{x})$$
(24)

In momentum space the continuum eqn(22) reads

$$(i\gamma_{\mu}p_{\mu} + m)\psi = 0 \tag{25}$$

while the lattice-Dirac eqn (23) reads

$$(i\gamma_{\mu}\sin p_{\mu} + m)\psi = 0 \tag{26}$$

As was first pointed out by Smit and Wilson(independently), the difference between the two Dirac eqns is profound. To see this note that the sin-function vanishes not only at $p_{\mu} \simeq 0$ but also at $p_{\mu} \simeq \pi_{\mu}^{A}$ where π_{μ}^{A} are four momenta such that the components are either 0 or π (note that p_{μ} , m in eqn(25,26) are dimensionless). Indeed, A takes values 1, ..., 16 corresponding to four vectors $(0,0,0,0),(\pi,0,0,0)(4$ in number), $(\pi,\pi,0,0)(6$ in number), $(\pi,\pi,\pi,0)$ (4 in number) and (π,π,π,π) . If we expand p_{μ} around π_{μ}^{A} as $p_{\mu} = \pi_{\mu}^{A} + q_{\mu}$ where q_{μ} are small, we would have $\sin p_{\mu} = \pm \sin q_{\mu}$. It is quite easy to find nonsingular operators S_{A} such that

$$S_A \gamma_\mu S_A^{-1} = \pm \gamma_\mu \tag{27}$$

for every μ such that $\pi_{\mu}^{A} = \pi$. In fact for any ρ such that $\pi_{\rho}^{A} \neq \pi, S_{A}$ can be chosen to be γ_{ρ} . Now the lattice Dirac eqn takes the form

$$S_A(i\gamma_\mu \sin q_\mu + m)S_A^{-1}\psi = 0$$
 (28)

for every value of A.Alternatively, $\tilde{\psi}_{(A)} = S_A^{-1} \psi$ satisfy

$$(i\gamma_{\mu}\sin q_{\mu} + m)\tilde{\psi}_{(A)} = 0 \tag{29}$$

Thus the lattice-Dirac eqn(23) actually represents 16 Dirac particles and in the continuum limit, the lattice theory has the **wrong** spectrum!

6 Vector Gauge Theories on The Lattice

Consider a Vector gauge theory like Quantum Chromodynamics(QCD) on the lattice. This is vector like because the left and right-handed fields both transform as the fundamental representation of SU(3). The SU(3) is also gauged, with the interaction between the gauge fields A^a_{μ} and the quark fields given by

$$\mathcal{L}_{fer,gauge} = A^a_\mu (\bar{\psi}_L \gamma^\mu \tau^a \psi_L + \bar{\psi}_R \gamma^\mu \tau^a \psi_R) \tag{30}$$

Wilson proposed the following remedy for the problem of "species doubling" (16 Dirac particles in place of 1, doubling in each space-time direction). He proposed modifying the fermion lagrangean to

$$\mathcal{L} = \frac{1}{2} \sum \bar{\psi} \gamma_{\mu} D_{\mu} \psi + m \sum \bar{\psi} \psi - r \bar{\psi} D_{\mu} D^{\mu} \psi$$
(31)

where

$$D_{\mu} = U_{\mu}(x)\psi(x + \mathbf{e}_{\mu}) - \psi(x) \tag{32}$$

with $U_{\mu}(x)$ being the link variables (see McKellar's talk for details). Wilson's modification has the effect of the replacement

$$\gamma^{\mu} \sin p_{\mu} \rightarrow \gamma^{\mu} \sin p_{\mu} + r \sum_{\mu} (1 - \cos p_{\mu})$$
 (33)

in the lattice-Dirac eqn(23).For $p_{\mu} \simeq \pi_{\mu}^{A}$, $A \neq 0$, the added terms are (2r, 4r, 6r, 8r). Thus for $r \neq 0$ they have the effect of moving the masses of the doublers to $\simeq \frac{1}{a}$ and hence infinity in the continuum limit. Consequently the doublers can be made to decouple in the continuum limit.

It is very important for this construction that the (added)Wilson term is **manifestly gauge-invariant**.

6.1 Anomaly on The Lattice

It is straight-forward to work out the divergence of the axial current in the lattice regularisation. The result is

$$\Delta_{\mu} j_{\mu}^{5}(x) = 2m\bar{\psi}(x)\gamma_{5}\psi(x) + \frac{r}{a}(\bar{\psi}(x)\gamma_{5}\psi(x) - \frac{1}{2}\sum_{\mu}\bar{\psi}(x)\gamma_{5}U_{\mu}\psi(x+\mathbf{e}_{\mu})) \quad (34)$$

When r = 0, i.e when the Wilson modification is not made, the anomaly is seen to vanish exactly (we will explain the physical origin of this shortly) but when $r \neq 0$ it can be shown that in the continuum limit $a \rightarrow 0$, the correct anomaly is reproduced by the above equation.

7 Chiral Fermions On The Lattice

Species doubling, unlike in the case of vector gauge theories where it can be handled quite satisfactorily, becomes really problematic in the case of chiral gauge theories and it essentially makes it very difficult to lattice-regularise such theories. However, very recently, a ray of hope appears to have emerged which will be discussed in the last section. Suppose we atart with

$$\mathcal{L}_{chiral} = \frac{1}{2} \sum \bar{\psi}_L \gamma_\mu D_\mu \psi_L \tag{35}$$

where $\psi_L = \frac{1}{2}(1 + \gamma_5)\psi$ is the left-handed field in the continuum. The naive expectation would be that the above lagrangean represents a single species of a chiral(left-handed in this case) fermion.REcall that in the case of the Dirac fermion,the naive expectation was belied by species doubling.What happens in the present case?Are there also 16 chiral(left-handed) fermions?In fact what happens is far more subtle and dangerous.

To see this recall that the transformation S_A used to map the lattice-Dirac eqn to the same form around all the π^A_μ changed the signs of those γ_μ such that μ were the directions where the components of π^A were π . This can be used to see the effect of S_A on $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. This is presented below

$$\Pi^{0} = (0,0,0,0) \ 1 +, +, +, + \gamma_{5} \to \gamma_{5}
\Pi^{\mu} = (\pi,0,0,0) \ 4 -, +, +, + \gamma_{5} \to -\gamma_{5}
\Pi^{\mu\nu} = (\pi,\pi,0,0) \ 6 -, -, +, + \gamma_{5} \to \gamma_{5}
\Pi^{\mu\nu\rho} = (\pi,\pi,\pi,0) \ 4 -, -, -, + \gamma_{5} \to -\gamma_{5}
\Pi^{S} = (\pi,\pi,\pi,\pi) \ 1 -, -, -, - \gamma_{5} \to \gamma_{5}$$
(36)

Thus around the points π^0 , $\pi\mu\nu$ and π^S we do indeed have left-handed modes but at π^{μ} , $\pi^{\mu\nu\rho}$ we actually have right-handed modes! Taking into account the multiplicities of these points we see that eqn() naively thought to represent **one** left-handed field actually represents **16** fields **8** of which are lefthanded and **8** are right-handed!Not only have the species been doubled,the naive **parity asymmetric** situation actually represents **parity symmetric** situation!

A more careful analysis reveals that the left and right-handed fields carry the same representation of the gauge group. Further the doubling arranges for the total axial charge to be zero i.e $\sum Q_5 = 0$.

Before analysing the implications of this striking result, let us explain the earlier mentioned result in the context of vector theories, namely, the vanishing of the anomaly in the r = 0 case. The naive Dirac field can again be thought

of as being composed of a right-handed field and a left-handed field, both transforming identically under the gauge group. As we have just seen, both the naive left and right-handed fields are really 8 left and 8 right-handed fields on the lattice. The total axial chrge being zero, there is no anomaly!

7.1 Lattice Dirac Sea Picture

At this stage it is instructive to understand the results of species doubling and the vanishing of the anomaly from a Lattice Dirac Sea picture. This was done by Ambjorn, Greensite and Peterson [3] by extending the Dirac Sea ideas of Nielsen & Ninomiya, and Peskin.

As in the continuum case, it is useful to adopt the Hamiltonian version. The lattice dispersion relation for Weyl particles reads

$$\omega_k = \pm \sum_i \sin k_i \tag{37}$$

This dispersion relation shows a dramatic change in the nature of the regularised Dirac sea! A naive regularisation of the Dirac sea would have envisaged a sea with a bottom but nevertheless such that the bottom is at a considerable depth. Consequently one may have imagined that the happenings at the bottom of the sea are not of much relevance to low energy phenomena. However, the lattice dispersion relation () shows that the states



p:1 0:1

Figure 5: The Lattice Dirac Sea

with maximum momentum in any of the directions(but zero momentum in the orthogonal directions) are also at **zero** energy and hence at the **top** of the sea! What happens there is very much of consequence for low energy physics!

Other notable features of the lattice Dirac sea are (i) no gauge invariance violation at the "bottom" of the sea and as discussed in the previous section,(ii) chirality is flipped at half the number of "bottoms".Putting all these together one finds that there is net pumping of chirality.

7.2 Generic nature of Doubling

One may wonder whether the species doubling that we have encountered is an artefact of the way the fermions have been latticised and whether with some luck one may find a way of latticising that would avoid species doubling. The answer to this as given by the first nielsen-Ninomiya theorem is NO.According to this theorem the occurrence of doubling is **generic**. The



Figure 6: A Function With a Pair of Simple Zeroes on The Circle

crux of this theorem is that the origin of species doubling is topological in nature. As already stated before the momentum space(Brilloin Zone) is a dtorus with Euler characterstic 0. The momentum space is also compact. This implies that simple zeroes of a function i.e zeroes near which the function is linear, must occur in pairs. It is the occurrence of such zeroes in pairs that translates into species doubling. Thus chirality also must occur in pairs of opposite chirality.

Thus the problem of finding a discretisation that avoids species doubling amounts to finding one whose associated Brilloin Zone has non-zero Euler Characterstic. This appears to be a very difficult task.

8 Standard Model on the Lattice?

If the standard model could also be formulated on the lattice, we would have a non-perturbative gauge-invariant formulation of it.But the lesson we have learnt is that species doubling makes the theory vector-like without parity violation, unless a clever way is found to avoid species doubling. This raises the following important question:

Can we move the doublers to the cut-off scale as was done for QCD by the Wilson Method? In the standard model the left-handed and the right-handed fields transform differently under the gauge group. This precludes a bare mass-term for the fermions. In fact, as discussed right in the beginning, masses for fermions are obtained through the Higg's mechanism.

For the same reasons, the Wilson mass term is also not gauge invariant in this context. There have been many attempts to formulate chiral gauge theories on tilhe lattice over the last 15 years. Most of them have failed in realising their objectives. In the next section I'll describe three attempts at solving this problem. It is very difficult to cover in a comprehensive way all the proposals that have been made to alleviate this problem. Many of the discussions in the literature are very technical. Often, a conceptual separation is lacking of the problem of putting generic chiral gauge theories on the lattice and the problem of putting the standard model on the lattice. Due to a lack of space and time, I have had to leave out the discussion of many interesting proposals like the Rome Propsal[7], the proposal of t'Hooft [5] to use different regularisations for fermions and gauge fields, the proposal of Slavnov [6], the overlap formalism of Narayanan and Neuberger [10] etc. (see [11] for a more detailed coverage)

9 Some Attempts To Put Standard-like Models On the Lattice

9.1 Wilson-Yukawa Models

[8] Let us again recapitulate how the "mass terms" for fermions were generated in a gauge invariant manner within the standard model: the mass term

$$\mathcal{L}_{mass} = m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{38}$$

is not gauge invariant in a theory where the left and right-handed fields transform differently under the gauge group. The remedy was to consider the gauge-invariant interaction term

$$\mathcal{L}_{Yukawa} = g_Y \bar{\psi}_L \Phi \psi_R + h.c \tag{39}$$

Gauge-invariance is achieved through a suitable transformation property of Φ . In the spontaneously broken phase of the theory, Φ develops a vacuum expectation value i.e $\langle \Phi \rangle = v$. Then the interaction term looks like

$$\mathcal{L}_{Yukawa} = (g_Y v) \bar{\psi}_L \psi_R + h.c + \dots \tag{40}$$

The fermion mass is given by $m_f = g_Y v$ and to get light fermions in the spectrum one has to tune g_Y appropriately.

One may attempt a similar trick to give masses of the order of cut-off to the unwanted doublers. The idea is to generalise the Wilson mass term into a gauge invariant Wilson-Yukawa term:

$$\mathcal{L}_{Wilson-Yukawa} = \bar{\psi}_L \Phi(y - w \sum_{\mu} \partial_{\mu} \tilde{\partial}_{\mu}) \psi_R + h.c$$
(41)

Here w plays a role similar to r in the Wilson term for vector theories; ∂_{μ} is the backward shift operator. It should be remarked that this construction is aimed more towards standard-model like theories rather than towards, say, chiral gauge theories on their own.

Now the doubler masses can be moved to infinity(cut-off scale) by taking the limit $w \to \infty$. But this puts the theory in the strongly coupled phase. A very careful study of this phase has been made by Golterman, Petcher, Smit and others (for details, see [8]). Their conclusions are as follows:

i)Gauge singlets are formed as bound states of ψ_L and Φ . This in itself need not be alarming in view of the so called t'Hooft Complimentarity picture according to which there is no phase boundary between the Higgs and confining phases. The spontaneously broken phase can be viewed in a manifestly gauge-invariant picture and the massive gauge bosons would be viewed more like the gauge-invariant glue ball states of pure QCD and would appear as bound states of the Higgs and gauge fields. But in the context of chiral fermions the situation could be potentially problematic as some authors claim a violation of complimentarity in this case. ii) The real problem for the construction comes when one examines the interactions in the theory. All interactions are seen to vanish in the $w \to \infty$ limit.

iii) The doublers and the Right-handed particles decouple leaving behind a massive fermion of mass $m_f = y$. Recall that $m_f = m_f^{phys}a$. Thus to get light fermions, y has to be extemely fine-tuned. In fact y has to vanish as a. The degree of fine-tuning needed is much more severe than what is required to get light fermions in the continuum standard model where $m_f = g_Y v$ because there $g_Y = m_f^{phys}/v^{phys}$.

With even very slight mismatch, all fermions decouple from the spectrum!. Even if fine-tuning could be achieved, the massless spectrum would consist of both right-handed and left-handed particles.

Thus the Wilson-Yukawa approach does not work.

9.2 Domain Wall Fermions

[9] Another interesting proposal to put chiral fermions on the lattice was put forward by Kaplan [9]. His proposal consists in working on a five dimensional lattice to start with. Since there are no chiral fermions in odd space-time dimensions and since there are no problems in formulating vector theories on the lattice, this five dimensional theory can be consistently and non-perturbatively formulated. Next he considers a four-dimensional domain



Figure 7: An Anti-Domain Wall

wall to which chiral fermions are constructed to be glued onto. The way this is accomplished is by considering a 5-dimensional Dirac fermion whose mass depends on the 5th coordinate as follows:

$$m_5(x^5) = m \quad x^5 > 0$$

$$= 0 \quad x^5 = 0 = -m \quad x^5 < 0$$
 (42)

Here m > 0 and the four-dimensional domain wall is at $x^5 = 0$.

The point is that normalisable solution of the Dirac equation is a **single** chiral fermion with $\gamma_5 = 1$ living on the domain wall.in the limit that the extent L_5 along the 5th direction tends to infinity. Narayanan and Neuberger have shown that this picture can be given a purely four-dimensional interpretation also. The anomaly in this picture arises as a Chern-Simons current flowing out of the domain wall.

9.2.1 Problems

With periodic boundary conditions (usually preferred in lattice studies) in the 5th direction, one inevitably has an **anti-domain wall** with a $\gamma_5 = -1$ chiral fermion living on it. For finite values of L_5 there are contaminations by unwanted chirality states. Coupling a gauge field to the domain wall chiral fermion requires d = 4 gauge fields only close to the domain wall and zero elsewhere. Though there are some proposals on how to handle this, the fact that this implies gauge invariance violation is not very encouraging.

It is of course possible to consider open boundary conditions in which case some of these problems disappear.What results then is essentially the overlap formalism of Narayanan and Neuberger.We shall not discuss that any further here as it is rather technical and the program is still incomplete.

9.3 Wilson-Ginsparg Method

Ginsparg and Wilson (for details see [13]) gave a very interesting interpretation of what exact chiral symmetry on the lattice means. They started with a cut-off theory with exact chiral symmetry and consider block spinning transformations which are chirally asymmetric. This way they obtain a coarse grained theory whose action is chirally asymmetric but whose continuum theory is indeed chirally symmetric. In this manner they found that the Grren's function of the coarse grained theory should satisfy

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D \tag{43}$$

instead of the naive symmetric Green's function that would satisfy $\gamma_5 D + D\gamma_5 = 0$. Incorporating non-abelian gauge fields can be problematic. INterest

in this idea was revived by the observation of Hasenfratz that the fixed point action of QCD satisfies the Ginsparg-Wilson relation.

Though the original work of Ginsparg and Wilson did not directly confront the problem of regularising chiral gauge theories, much attention has been focussed in that direction by an observation of Neuberger and Narayanan that the overlap formalism produces a D satisfying the Ginsparg-Wilson relation and by Lueschers [?] claim of having constructed a U(1) chiral gauge theory on the lattice. On introducing the "lattice chiral transformations"

$$\delta\psi = \gamma_5 (1 - \frac{aD}{2})\psi \tag{44}$$

$$\delta\bar{\psi} = \bar{\psi}(1 - \frac{aD}{2})\gamma_5 \tag{45}$$

for abelian gauge theories and

$$\delta\psi = T\gamma_5(1 - \frac{aD}{2})\psi \tag{46}$$

$$\delta\bar{\psi} = \bar{\psi}(1 - \frac{aD}{2})\gamma_5 T \tag{47}$$

for non-Abelian gauge transformations, it is easy to verify that

$$\delta(\bar{\psi}D\psi) = 0 \tag{48}$$

However, the measure for the functional integration over fermions is not invariant under these transformations and the anomalies are reproduced this way very much the way the Fujikawa derivation of anomalies works in the continuum.

Though the Ginsparg-Wilson method offers at the moment the best hope for putting chiral gauge theories on the lattice, there are still many open issues. Even Lueschers[?] construction is a progress as far as matters of principle are concerned, but is not at a stage where one can implement it. The non-Abelian extensions of it, an understanding of the GW construction in terms of the lattice Dirac sea, a better understanding of how the Nielsen-Ninomiya theorem is circumvented are issues yet to be tackled. The eventual goal would of course be an implementable lattice regularisation of the standard model.

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