Foundations of Lattice Field Theories-2

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Continuum Limit of Lattice Field Theories

- As emphasized before lattices in this context are fundamental unlike in condensed matter.
- a^{-1} acts like the Ultraviolet cut-off and must eventually be taken to ∞ i.e. $a \to 0$
- Hopefully in this limit all memories of the lattice are removed.
- non-trivial! Taxi-driver metric. $d = |x| + |y|$
- Lorentz and Rotational invariances should be restored.
- Roughening Transition

How cutoff is removed in continuum perturbation theory?

- Let the bare parameters of the theory be χ_0.
- Calculations end up in divergences due to short-distance singularities.
- Regularize the theory with cut off Λ.
- Calculations are finite modulo Infrared Divergences
- Renormalizability of the theory is not relevant at this stage.
- Even a non-renormalizable theory like gravitation has a well defined regularized version.

Renormalisation in continuum perturbation theory?

- Let G be some observable, mass, charge, S-matrix ...
- In the regularized theory $G(\chi_0, \Lambda, P..)$
- At this stage, $G(\chi_0, \Lambda..) \to \Lambda^b + \ldots$ for some positive b
- Cut off can not be removed at this stage.
- At this stage one introduces a dependence between the bare parameters χ_0 and the cut-off Λ by fixing some particular G, say, $\bar{G}$. Renormalisation.

$$\bar{G}(\chi_0, \Lambda) = G_{phys}$$

- This induces renormalised parameters $\chi_r(\chi_0, \Lambda)$-fixed. The observables can now be cast as $G(\chi_r, \Lambda)$
- What is gained?

$$G(\chi_r, \Lambda) \to \Lambda^0 + \Lambda^{-c} + \ldots$$

for large Λ.

Renormalisation in continuum perturbation theory?

- Caution: there is still cut off dependence even after renormalisation!. But it is much softer and allows for the cut off to be removed slowly.
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Removing the cutoff in lattice theories

- The same strategy should work on the lattice too.
- But the cut-off(lattice spacing) has disappeared altogether!
- One uses a different strategy: the ratio of any physical length scale to a should become infinite!
- This physical scale in lattice units translates to a lattice correlation length
- Hence the CSM of lattice theories must have phase transitions with diverging correlation lengths.
- Highly non-trivial

Phase transitions and Critical points of CSM: Landau Scheme

- The Free energy $F(\beta)$ is given by $Z(\beta) = e^{-F(\beta}$
- $F(\beta)$ is a continuous function of β
- But it's derivatives need not be continuous everywhere nor even exist
- First derivative discontinuous – First order phase transitions. Correlation lengths do not diverge.
- Ice-Water, Water-Steam...
- The critical end-point of water-steam has diverging correlation functions.
- Correlation length divergence governed by critical exponents
- Large class of CSM's have the same critical exponents even if their 'physics' is very different universality

Statistical Continuum Limit

- I now give a toy model in which we can calculate the correlation lengths exactly.
- It uses the concept of Transfer Matrices that I already talked about.
- The toy model is the $D = 1$ lattice field theory I introduced earlier:

$$Z = \int \prod d\phi_n \, e^{-(2+m^2)\sum \phi_n^2 + \sum \phi_n \phi_{n+1}}$$

- Introduce a 'temperature', actually $\beta = \frac{1}{kT}$ as follows

$$Z(\beta) = \int \prod d\phi_n \, e^{-(2+m^2)\sum \phi_n^2 + \beta\, \phi_n \phi_{n+1}}$$

- The significance of the special value $\beta_c = 2$ will become clearer shortly.
- The Correlation Functions of this model are given by

$$\langle \phi_1 \dots \phi_n \rangle = Z(\beta)^{-1} \int \prod d\phi_n \, \phi_1 \dots \phi_n \, e^{\cdots}$$

The Transfer Matrix

- we now write down the Transfer Matrix for this system(call $2 + m^2 = A$)

$$T[\phi_{n+1}, \phi_n] \;=\; e^{-\frac{A}{2}(\phi_{n+1}^2 + \phi_n^2) + \beta\, \phi_n \phi_{n+1}}$$

- To avoid various subtle but inessential issues, we take the $D = 1$ 'space-time' manifold be to be a very large circle with N points, eventually letting $N \to \infty$.
- Then it is clear that

$$Z(\beta) = \int \prod d\phi_n \, T[1,2] T[2,3] \ldots \; T[N-1, N] = TrT^N$$

- Let $\Lambda_0, \Lambda_1 \ldots$ be the eigenvalues of the infinite dimensional Transfer matrix, arranged in decreasing order.
- Since $Tr\, T^N = \sum_i g(i)\Lambda_i^N$ and we are eventually going to take $N \to \infty$, only the largest eigenvalue will dominate

$$Z(\beta) = \Lambda_0^N$$

- The eigenvalue eqn:

$$T\Psi_0 = \Lambda_0\,\Psi_0 \to \int d\phi_n\, T[\phi_{n+1}, \phi_n]\Psi_0(\phi_n) \;=\; \Lambda_0\,\Psi_0(\phi_{n+1})$$

- All integrals are Gaussian
- In fact, all eigenstates are Harmonic Oscillator wavefunctions:

$$\Psi_0(\phi_n) \;=\; \frac{\alpha}{\pi}^{\frac{1}{4}} e^{-\frac{\alpha}{2}\phi_n^2} \quad \alpha = \sqrt{A^2 - \beta^2} \quad \Lambda_0 = \sqrt{\frac{2\pi}{A+\alpha}}$$

First Excited state

- We will now calculate the the first excited state with eigenvalue $\Lambda_1 \leq \Lambda_0$.

$$T\Psi_1 = \Lambda_1 \Psi_1 \rightarrow \int d\phi_n \, T[\phi_{n+1}, \phi_n]\Psi_1(\phi_n) = \Lambda_0 \Psi_1(\phi_{n+1})$$

- With the condition $\langle \Psi_1 | \Psi_0 \rangle = 0$
- These too are easily solved:

$$\Psi_1(\phi_n) = \sqrt{2\alpha}\, \phi_n \Psi_0(\phi_n) \qquad \Lambda_1 = \frac{\beta}{A+\alpha} \Lambda_0$$

- Little algebra shows $\Lambda_1 \leq \Lambda_0$.
- Exercise: Find all the eigenstates and eigenvalues
- The transfer matrix is Positive. The positive Hilbert space of QM.

Two point function

- It follows:

$$\langle \phi_n \phi_{n+R} \rangle = c_n c_{n+R} \left(\frac{\Lambda_1}{\Lambda_0}\right)^R + \dots$$

- When R is very large, only this term dominates:

$$\langle \phi_n \phi_{n+R} \rangle = c_n c_{n+R} e^{-\frac{R}{\xi}} \qquad \frac{1}{\xi} = -\ln \frac{\Lambda_1}{\Lambda_0}$$

- ξ is the Correlation Length for this channel.
- It clearly diverges when $\Lambda_1 = \Lambda_0$.
- This toy model actually has a phase transition with diverging correlation length!(the stat mech folklore says there can be no phase transitions in $D = 1$!!)

The Critical Point of our $D = 1$ theory.

- The critical point β_c is given by the condition
- This is where the lattice mass gap vanishes.
- Writing out in detail

$$\beta_c = A + \sqrt{A^2 - \beta_c^2}$$

- Since this amounts to the lattice spacing $a \to 0$, in leading order m^2 can be set to 0, yielding

$$\beta_c \approx A \approx 2$$

- Explaining the special value of $\beta = 2$.

Critical Exponents of the model

- It is straightforward to show that

$$\xi(\beta) \to (\beta - \beta_c)^{-\frac{1}{2}} \qquad \beta \to \beta_c = 2$$

- The critical exponent is $\frac{1}{2}$ Mean Field Exponent
- Equivalently $a \approx (\beta - 2)^{\frac{1}{2}}$
- Exercise: Find all the correlation lengths
- Do all of them diverge at the same β?
- How many of them are independent?

Back to the latice

- So renormalisation on the lattice amounts to keeping some physics constant.
- There may be many sets of diverging correlation lengths offering physically distinct continuum limit
- Or different correlation lengths diverging at the same critical point but proportional to each other

$$m(\beta) = m_{phys} \cdot a \rightarrow a = \frac{1}{m_{phys}} \cdot \frac{1}{\xi(\beta)}$$

- $\frac{m^1_{phy}}{m^2_{phy}} = \frac{m^1(\beta)}{m^2(\beta)}$ only very close to β_c
- In Abelian Higgs model in $D = 3$, string tension and mass gap have independependent correlation lengths.