Foundations of Lattice Field Theories-2

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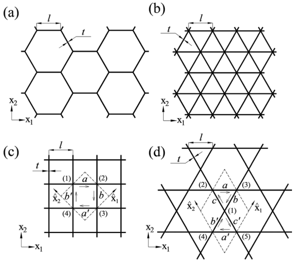
Three Pillars: Discretisation



- The third pillar consists in discretising space-time continuum.
- One first replaces it by a discrete set of points called a lattice
- The lattice can be freely chosen. Some choices for D=2 are shown above
- Often the hypercubic lattices are chosen for their simplicity.

Three Pillars: Discretisation

- One then replaces derivatives in the continuum Euclidean action by appropriate finite differences.
- To illustrate, let us consider a $D = 1$(space-time dim) 'Field Theory'. This is actually QM of one degree of freedom $x(t)$.
- The Euclidean action can be taken to mimic a massive scalar field theory:

$$S_E = \int dx[(\partial_x\phi)^2 + m^2\phi^2]$$

- The lattice can be chosen to be the set of equally spaced points along a line with spacing, say,a. $x_n = n \cdot a$
- Even a random set of points along a line can be a bona fide choice.

Discretisation

- The next step is replacing all derivatives of fields by finite differences. There are many choices possible

$$\partial_x \phi \rightarrow \frac{\phi(x+a)-\phi(x)}{a}, \frac{\phi(x)-\phi(x-a)}{a} \ldots$$

- Each choice leads to a different lattice action.
- Let us explicitly work out the first choice:

$$S_E \rightarrow S^L = a \sum_n \frac{(\phi_{n+1}-\phi_n)^2}{a^2} + m^2 a \sum_n \phi_n^2 \qquad \phi_n \equiv \phi(x_n)$$

- Since $\frac{S_E}{\hbar}$ is dimensionless, S^L is also dimensionless. Further, one scales all quantities by suitable powers of a to make them dimensionless.
- $m^L = m \cdot a$. The mass dimension of a scalar field in D dimensions is $\frac{D-2}{2}$. So $\phi^L = \phi \cdot a^{-1/2}$.

Discretisation

- It is easy to see that S^L takes the form

$$S^L = 2\sum_n [(\phi_n^L)^2 - \phi_n^L \phi_{n+1}^L] + m_L^2 \sum_n (\phi_n^L)^2)$$

- We will drop the superscript L henceforth.
- The path-integral now becomes

$$Z = \int \prod_n d\phi_n e^{-(2+m^2)\sum_n \phi_n^2 + 2\sum_n \phi_n \phi_{n+1}}$$

- As before it is always understood that one should be computing $Z(J)$

Discretisation

- This is a classical statistical mechanics partition function for a one-dimensional chain of nearest-neighbour interacting 'continuous' spins x_n.
- Though we established this deep connection for an almost trivial example, this mapping is generic.
- D(space-time) QFT is exactly mappable to a problem in classical statistical mechanics problem in D spatial dimensions!
- Allows a whole body of powerful analytical and numerical techniques of CSM to investigate QFT's
- See Seiler's monograph for an extensive coverage of how techniques of CSM can be used to address such issues as phase boundaries between the Higgs and Coulomb phases etc..

- Let us now consider Abelian gauge fields(electromagnetism) in Minkowski space.
- We shall restrict ourselves to pure gauge case. The action in $D = 4$ is

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu \,-\, \partial_\nu A_\mu$$

- Euclideanising via $x^0 \to\ -ix_4, A_0 \to\ iA_4$

$$S_E = \int d^4x\, F_{\mu\nu}F_{\mu\nu}\ \geq\ 0$$

- The original action was invariant under

$$A_\mu \to\ A_\mu + \partial_\mu \Lambda(x)$$

Discretisation of Abelian Gauge Theories

- We need to find discretised forms of both the action and the gauge transformation.
- Introduce the forward derivative

$$\Delta_\mu f(x) = f(x + a e_\mu) - f(x)$$

- With its help we can discretise the $F_{\mu\nu}$:

$$F_{\mu\nu} \to \frac{1}{a}(\Delta_\mu A_\nu - \Delta_\nu A_\mu)$$

- One choice for the discretised gauge transformations is

$$A_\mu \to A_\mu + \frac{1}{a}\Delta_\mu \Lambda(x)$$

- It is easily verified that the discretised action is invariant under the discretised gauge transformations.

Discretisation of Abelian Gauge Theories

- As before, all dimensionfull quantities are scaled by suitable powers of the lattice spacing a to get their dimensionless versions:

$$A^L_\mu = a \cdot A_\mu \qquad F^L_{\mu\nu} = a^2 \cdot F_{\mu\nu}$$

- Then

$$F^L_{\mu\nu} = \Delta_\mu A^L_\nu - \Delta_\nu A^l_\mu \qquad A^L_\mu \to A^L_\mu + \Delta_\mu \Lambda(x)$$

- The resulting abelian lattice gauge theory is

$$Z = \int \prod_{\mathbf{n},\mu} dA_{\mathbf{n},\mu}\, e^{-\sum F^2_{\mu\nu}}$$

- We will encounter a very different discretisation of Abelian gauge theories shortly. This one can be called an Lie-algebra discretisation. Also called non-compact case
- Configurations differing by gauge transformations make the same contribution and there are infinitely many leading to a divergence of Z. Gauge-fixing.

Non-Abelian Gauge Theories

- Now we turn to a discussion of Non-abelian gauge theories on the lattice.
- We shall consider the so called compact Lie Groups like SU(N) or U(N).
- dim(G) is the dimension of the group. For SU(2) it is 3, for SU(3) it is 8...
- Instead of a single A_μ of the abelian case we now have as many A_μ's as dim(G): A_μ^a where the group index a takes on dim(G) values.
- The Hermitian generators L_a satisfy the Lie-algebra

$$[L_a, L_b] = i\, f_{ab}^c\, L_c \qquad Tr_f(L_a L_b) = \frac{1}{2}\,\delta_{ab}$$

- f_{ab}^c are the structure constants of the Lie Algebra.
- It is more convenient to use the Lie-algebra valued $\mathbf{A}_\mu \;=\; L_a\, A_\mu^a$.

Non-Abelian Gauge Theories

- The non-abelian gauge transformation:

$$\mathbf{A}'_\mu = g(x)\mathbf{A}_\mu\, g^\dagger(x) \;-\; \frac{i}{g_0}\, g(x)\partial_\mu\, g(x)^\dagger$$

- The $g(x)$ are group elements for each x. They play the role of $\Lambda(x)$ of the abelian case.
- The Lie-algebra valued field strengths are given by

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu \;-\; \partial_\nu \mathbf{A}_\mu \;+\; ig_0[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

- Under the gauge transformations the field strenths transform as

$$\mathbf{F}'_{\mu\nu} \;=\; g(x)\mathbf{F}_{\mu\nu} g(x)^\dagger$$

- The field strengths, unlike the abelian case, are not invariant but only transform covariantly

Non-Abelian Gauge Theories

- Let us attempt to discretise the non-abelian gauge field theory(Euclideanisation assumed).
- If we just mimick what we did in the abelian case:

$$\mathbf{F}_{\mu\nu} = \Delta_\mu \mathbf{A}_\mu \; - \; \Delta_\nu \mathbf{A}_\nu \; + \; ig_0[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

and

$$\mathbf{A}'_\mu = g(x)\mathbf{A}_\mu \, g(x)^\dagger \; - \; \frac{i}{g_0} g(x) \frac{g^\dagger(x+ae_\mu) \; - \; g^\dagger(x)}{a}$$

- The field strengths do not transform correctly!
- This is not due to the deficiency of this particular choice - no matter what one does, as long as the variables are $\mathbf{A}_\mu$, the problem can not be fixed!

Non-Abelian Gauge Theories

- The resolution rests on some fundamental mathematical properties of group connections!
- Of fundamental importance to our goal is the concept of holonomy:

$$U_{\mathbf{x},\mathbf{y}} \equiv P\, e^{ig_0 \int_{\mathbf{x}}^{\mathbf{y}} \mathbf{A}_\mu \, dx^\mu}$$

- The integration is along some specified curve Γ joining the two end points.
- The meaning of the path ordered exponential: divide the curve into a very large number of segments $xX_1, x_1 x_2, \ldots x_n y$. Then

$$P\, e^{\int_x^y} \equiv (e^{\int_x^{x_1}})(e^{\int_{x_1}^{x_2}}) \ldots (e^{\int_{x_n}^{y}})$$

- Most amazingly, the infinitesimal holonomies transform under gauge transformations as

$$U'_{\mathbf{x},\mathbf{x}'} = g(x)\, U_{\mathbf{x},\mathbf{x}'}\, g(x')^\dagger$$

- This, on using that each $g(x)$ satisfies $g(x)^\dagger\, g(x) = 1$, leads to the transformation law of all holonomies:

$$U'_{\mathbf{x},\mathbf{y}} = g(x) U_{\mathbf{x},\mathbf{y}} g(y)^\dagger$$

- Now the recipe for constructing non-abelian gauge theories on the lattice is to use holonomies along each link of the hypercubic lattice as the gauge field variable called Link Variables.

Compact vs Non-compact

- In the abelian case, the gauge fields were A_μ in the range $[-\infty, \infty]$ with total volume $\int dA_\mu = \infty$ – a noncompact space
- The gauge parameters were also in the same range with their space also being noncompact
- In the nonabelian case, the gauge parameters are the group elements $g(x)$
- Their total 'volume' is given by the group invariant Haar Measure:$\int dg(x) = 1$ – a compact space
- The gauge fields are also the link variables which are group elements with total volume $\int dU = 1$, also compact
- Both have to be compact or noncompact.

Invariants of non-abelian gauge theories:continuum

- The field strengths $\mathbf{F}_{\mu\nu}$ transforming as $F' = g\,F\,g^\dagger$ means FF, FFF all transform covariantly.
- Taking traces, e.g. Tr FF, using $gg^\dagger = 1$, are group invariants
- An invariant action:

$$\mathcal{L} = -\frac{1}{2}Tr\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu,a}$$

Invariants of non-abelian gauge theories:Lattice

- The Wegener-Wilson Loops: Take any closed loop built sequentially out of link variables

$$W = U_{1,2} U_{2,3} \ldots U_{n,1}$$

- Under gauge transformations $W' = gWg^\dagger$ and Tr W is gauge invariant.