Foundations of Lattice Field Theories.

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Some References

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- Regularisation of Chiral Gauge Theories - N.D.Hari Dass, Int.J.Mod.Phys B14, p.1989(2000); hep-th 9906158.
- Gauge Theories as a problem of constructive quantum field theory and statistical mechanics – Erhard Seiler, Springer Lecture Notes in Physics, 159(1982).

Why lattice field theories?

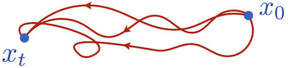
- On the Utilitarian side, numerical simulations have dramatically changed our understanding of QCD(confinement, monopoles, topology, spectrum..), non-perturbative Higgs phenomena etc..
- Incorporation of fermions, though very challenging, is rapidly approaching respectable levels..
- Even on the precision front, there has been impressive growth, for example recent muon $g - 2$..
- Still outstanding: Supersymmetric field theories
- Chiral gauge theories (the Standard model)
- Both having to do with conceptual as well as technical difficulties with Chiral Fermions.

Why lattice..

- On the conceptual side, QFT's are ill-defined because of the divergence difficulties.
- In the operator approach, singular behaviour of products of fields at a point..
- In the path-integral approach, giving meaning to the measure..
- Divergence of loop integrals in perturbation theory..
- While these challenges have been met quite adequately perturbatively, the QFT's have to be given a mathematically well-defined meaning non-perturbatively too
- Constructive Definition
- Lattice approach is the only known way of defining QFT's constructively
- Erhard's Seiler's monograph gives a very detailed and beautiful elaboration of this

The three pillars: path intyegrals

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



- The first of the pillars is the Path integral quantization
- The sum over all the paths with weights $e^{i\frac{S_{cl}}{\hbar}}$ yields the probability amplitude $K(x_i, t_i; x_f, t_f)$
- This is also formally represented as

$$K = \int \mathcal{D}X(t)\, e^{i\frac{S_{cl}}{\hbar}}$$

- A superficial mystery is how can the purely classical S_{cl} determine the entire quantum behaviour?
- Much of the answer depends on the measure.
- The folklore is that Feynman got the idea from some rudimentary considerations of Dirac about the role of action in quantum mechanics.

The three pillars: Path Integrals

- It turns out that Dirac had the full path-integral formulation!
- Of even more general kinds than the Feynman one!
- Dirac and the path integral arXiv:2003.12683

Path integrals..

- The Feynman methos works only when the Hamiltonian depends quadratically on p i.e. $H = \frac{p^2}{2m} + \ldots$
- Dirac eqn is linear in momentum!
- The generalisation to QFT's works similarly though the problem of the measure is harder
- Extension to gauge theories, both abelian and non-abelian also goes through
- Extension to fermionic field theories(Dirac eqn) is full of difficulties: $\psi, \bar{\psi}$ have to be treated as independent
- The Fermi statistics forces $\psi(\bar{\psi})$ to be described by anti-commuting Grassmann numbers
- N such numbers can still be represented by matrices but of size $2^N \times 2^N$!

Path integrals...

- They are formally like the partition functions of statistical mechanics
- The field theory path integrals have the interpretation of $\langle\, vac_- | vac_+ \rangle$
- The partition function in stat mech depends on the temperature $Z_{stat}(\beta)$, but the path integral has no parameters that it depends on!
- the really useful quantities are

$$Z(J) = \int \mathcal{D}\phi \, e^{i\frac{S_{cl}}{hbar}} \, e^{iJ\phi}$$

- These generate all the correlation functions $\langle\, vac_- | \phi \ldots \phi | vav_+ \rangle$
- All information about the QFT is codified in these correlations.
- This is the exact analog of the Wightman Functions $\langle\, 0 | T(\phi \ldots \phi) | 0 \rangle$ in Axiomatic Field Theory.

Path integrals...

- For bosonic field theories, they are of configuration type
- For fermionic field theories they are of the phase space type
- Most actions of interest involve fermion fields as bilinears and can be exactly integrated out, but result in determinants of matrices that grow exponentially with system size.
- Even if the fermion fields don't appear as bilinears, they can be brought to that type via bosonic auxilliary fields.
- Path integrals were formulated by Feynman in 1948.

The three pillars: Euclideanisation

- The next pillar is a truly mysterious aspect of QFT's called Euclideanisation
- Of course, one oftens euclideanises loop integrals but that is just for mathematical convenience.
- But now we will discuss euclideanisation of the entire QFT, not process by process.
- This is formulating the theories in a world with Euclidean metric $(+,+,+,+)$ instead of the real world Minkowski metric $(-,+,+,+)$
- There are dramatic physical differences between the two!
- In the Euclidean world, there is no time, causality etc, the very foundations of physics!

The three pillars: Euclideanisation

- To get some idea of why anybody would even want to attempt this, let us consider the simple cases of free propagators in minkowski space:

$$\frac{i}{p_0^2 - \mathbf{p}^2 - m^2 \pm i\epsilon}$$

- there are two complex propagators, though related.
- In contrast, the euclidean propagator

$$\frac{1}{p_4^2 + \mathbf{p}^2 + m^2}$$

  is a single real function!
- But how does this know of the two complex functions of the Minkowski theory?
- It is worth emphasizing that euclideanisation is logically a structure independent of path integrals or lattice etc.
- Now we delve into the origins of this deep idea.

The three pillars: Euclideanisation

- Schwinger introduced the The Euclidean Postulate in 1958, followed by Nakano in 1959.
- The amplitudes of a relativistic QFT continue to be meaningful under the mapping of Minkowski space onto Euclidean space
- A detailed correspondence can be established between RQFT in Minkowski space and a mathematical image based on a Euclidean manifold
- He expressed this in his rather picturesque way It is as if nature formulated her thoughts first in the Euclidean language and then rendered them for a Minkowskian world!
- More technically, the Euclidean postulate posits an exact correspondence between the Wightman functions and thev Schwinger Functions:

$$\langle \phi_E \ldots \phi_E \rangle$$

The three pillars: Euclideanisation

- Schwinger offered many interesting perspectives in support of his Euclidean postulate.
- Mathematicians had already known that some representations of the Lorentz Group could be obtained from the corresponding Euclidean Group representations through Weyl's Unitary Trick
- The Schwinger Euclidean postulate requires that ALL representations of the Lorentz Group that are of physical interest can be obtained that way.
- Generalizing what we noted about propagators, all Green's functions of the Minkowski theory are complex pairs coded by single real Euclidean Green's functions.

The three pillars: Euclideanisation

- It took another 15 years before Osterwalder and Schrader formulated their Axioms for Euclidean Green's functions in 1973.
- They formulated the precise conditions the Schwinger functions must satisfy in order they correspond to the Minkowski Green's functions.
- They introduced the powerful notion of Reflection Positivity as the conditions the Schwinger Functions must satisfy.
- It is Reflection Positivity that also guarantees that the corresponding Minkowski theory has a Positive-norm Hilbert space and Unitarity
- This is done through the concept of Transfer Matrices, a very powerful tool in statistical mechanics.
- Seiler's monograph is an excellent source for understanding these.
- Fermions pose difficulties as the Clifford Algebras are sensitive to both dimensions and the metric.

A boon from Euclideanisation

- The QFT path integrals, apart from a formal measure, also suffer from highly Oscillatory integrand, and that too in infinite dimensions!
- Euclideanisation cures this in a dramatic manner that is crucial for Lattice Field Theories.
- Let us first illustrate this for scalar field theories.
- $x^0 = ct \rightarrow -ix_4$.
- $ds_M^2 = -c^2 dt^2 + dx_i \cdot dx_i \rightarrow ds_E^2 = \sum_{i=1}^4 dx_i^2$
- The four-dimensional Minkowski Volume element $d^3x\, dt \rightarrow -i \prod dx_i$

A boon from Euclideanisation

- $$\mathcal{L} = \frac{1}{2}[(\partial_t\phi)^2 - \nabla\phi\cdot\nabla\phi] - V(\phi)$$
- $V(\phi) \geq 0$ for the Hamiltonian to be bounded.
- $$\mathcal{L}_E = -[\sum \partial_i\phi^2 + V(\phi)] \leq 0$$
- $iS_{cl} \to -S_E$ with $S_E \geq 0$
- The oscillatory $e^{iS_{cl}} \to e^{-S_E}$
- Same happens for gauge theories too as long as the time-like components are transformed as in x^0.

A big exception

- The Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi\, G} \int d^D x \sqrt{-g}\, R$$

- But the Euclidean scalar curvature R_E need not be of a definite sign!m
- A potential thorn in the flesh for path integral quantization of gravity!