

Quantum Chromodynamics

lecture III

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Plan

- Introduction to QCD
Monday, February 24, 2025
- QCD at work: infrared safety and jets
Tuesday, February 25, 2025
- *QCD at work: factorization and evolution*
Wednesday, February 26, 2025
- Deep structure of proton
Thursday, February 27, 2025

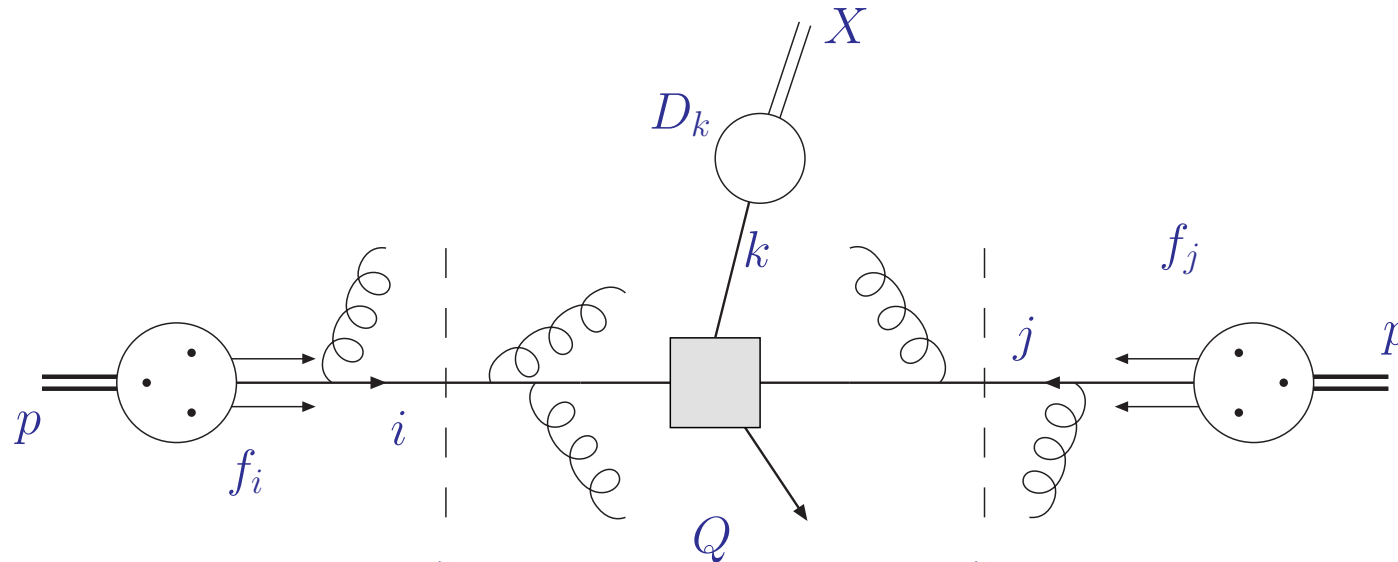
Factorization

- Large class of hard-scattering reactions with initial state hadrons
 - cross section not infrared safe
 - dependent on quark and gluon degrees of freedom in hadron
 - sensitive to nonperturbative processes at long distances
- Factorization of cross section
 - infrared safe hard part $\hat{\sigma}_{\text{pt}}$ calculable in perturbative QCD
 - nonperturbative function f determined from data
 - f parametrizes hadron structure
- General structure of cross section
 - large momentum scale Q , factorization scale μ

$$Q^2 \sigma_{\text{phys}}(Q) = \hat{\sigma}_{\text{pt}}(Q/\mu, \alpha_s(\mu)) \otimes f(\mu)$$

- convolution $\otimes$ in suitable kinematical variables
- Factorization
 - generalization of operator product expansion

QCD factorization

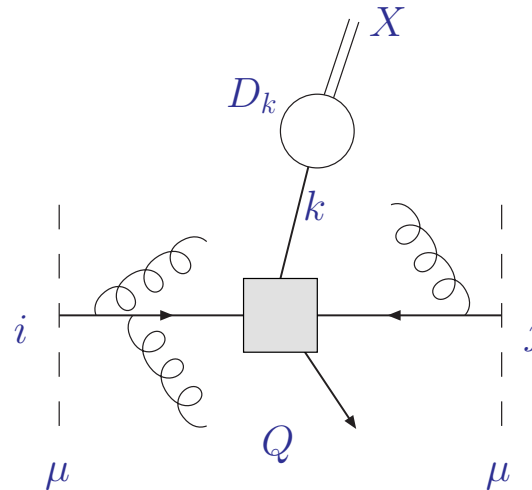


$$\sigma_{pp \rightarrow X} = \sum_{ij} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \hat{\sigma}_{ij \rightarrow X}(\alpha_s(\mu^2), Q^2, \mu^2, m_X^2)$$

- Factorization at scale μ
 - separation of sensitivity to dynamics from long and short distances
- Hard parton cross section $\hat{\sigma}_{ij \rightarrow X}$ calculable in perturbation theory
 - cross section $\hat{\sigma}_{ij \rightarrow k}$ for parton types i, j and hadronic final state X
- Non-perturbative parameters: parton distribution functions f_i , strong coupling α_s , particle masses m_X
 - known from global fits to exp. data, lattice computations, ...

Hard scattering cross section

- Parton cross section $\hat{\sigma}_{ij \rightarrow k}$ calculable perturbatively in powers of α_s
 - known to NLO, NNLO, ... ($\mathcal{O}(\text{few}\%)$ theory uncertainty)



- Accuracy of perturbative predictions
 - LO (leading order) ($\mathcal{O}(50 - 100\%)$ unc.)
 - NLO (next-to-leading order) ($\mathcal{O}(10 - 30\%)$ unc.)
 - NNLO (next-to-next-to-leading order) ($\lesssim \mathcal{O}(10\%)$ unc.)
 - N³LO (next-to-next-to-next-to-leading order)
 - ...

Parton luminosity

- Long distance dynamics due to proton structure



- Cross section depends on parton distributions f_i

$$\sigma_{pp \rightarrow X} = \sum_{ij} f_i(\mu^2) \otimes f_j(\mu^2) \otimes [\dots]$$

- Parton distributions known from global fits to exp. data
 - available fits accurate to NNLO
 - information on proton structure depends on kinematic coverage

Deep-inelastic scattering

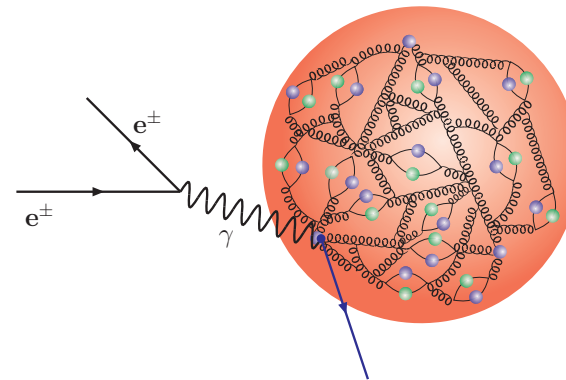
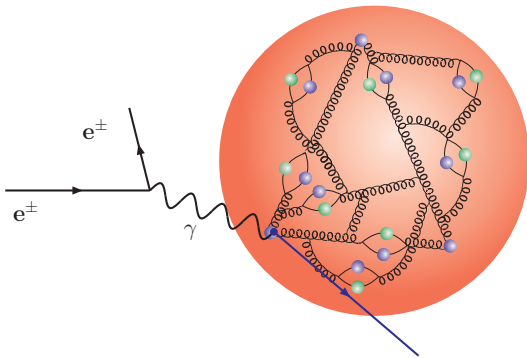
Classic example

- Deep-inelastic scattering
 - test parton dynamics at factorization scale μ

$$\sigma_{\gamma p \rightarrow X} = \sum_i f_i(\mu^2) \otimes \hat{\sigma}_{\gamma i \rightarrow X}(\alpha_s(\mu^2), Q^2, \mu^2)$$

Physics picture

- QCD factorization
 - constituent partons from proton interact at short distance
 - photon momentum $Q^2 = -q^2$, Bjorken's $x = Q^2/(2p \cdot q)$
 - low resolution
- high resolution



Once upon a time in the north . . .

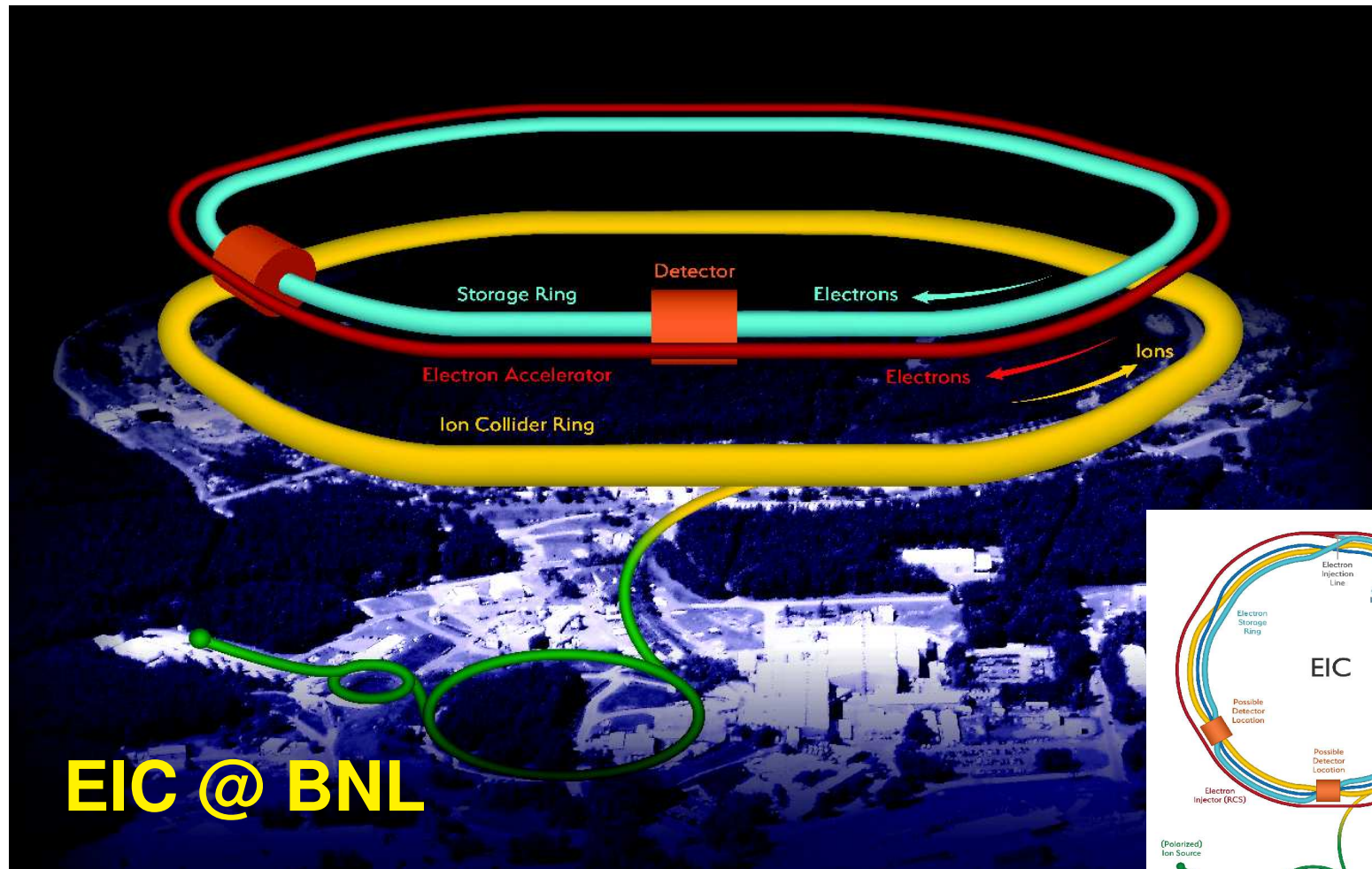
- HERA: deep structure of proton at highest Q^2 and smallest x



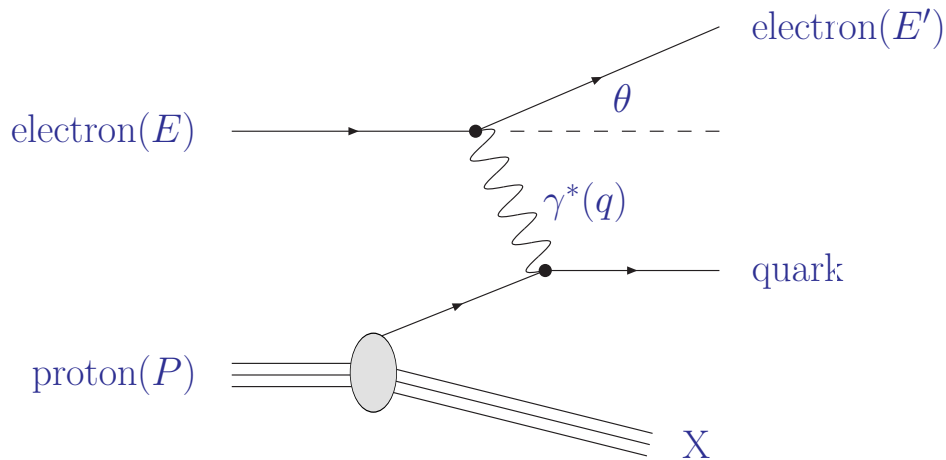
Bright future for precision hadron physics

- Electron-Ion Collider

A machine that will unlock the secrets of the strongest force in Nature



Inelastic electron-proton scattering



- Virtuality of photon: resolution

$$Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$$

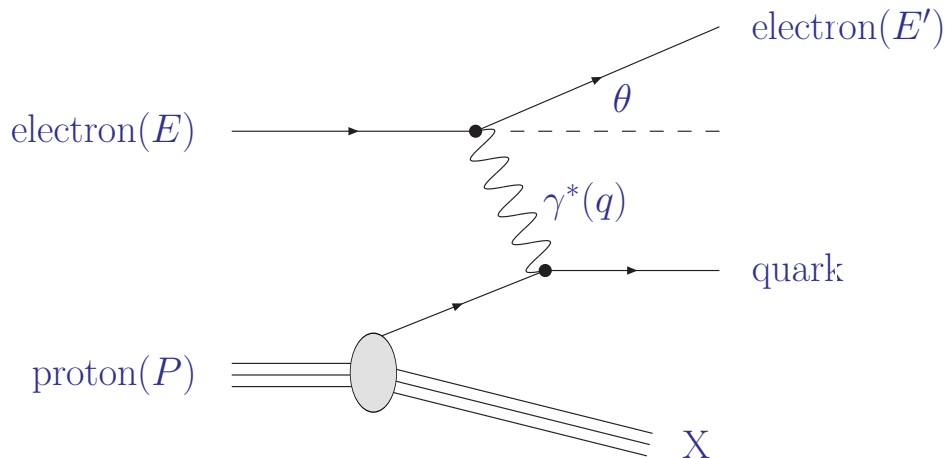
- Bjorken variable: inelasticity

$$x = \frac{Q^2}{2P \cdot q} < 1$$

- Cross section (X inclusive): proton structure function F_i^p

$$(E - E') \frac{d\sigma}{d\Omega dE'} \stackrel{\text{lab}}{=} \underbrace{\frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}}}_{\text{Mott-scattering (point-like)}} \left\{ F_2^p(x, Q^2) + \tan^2 \frac{\theta}{2} F_1^p(x, Q^2) \right\}$$

Inelastic electron-proton scattering



- Virtuality of photon: resolution

$$Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$$

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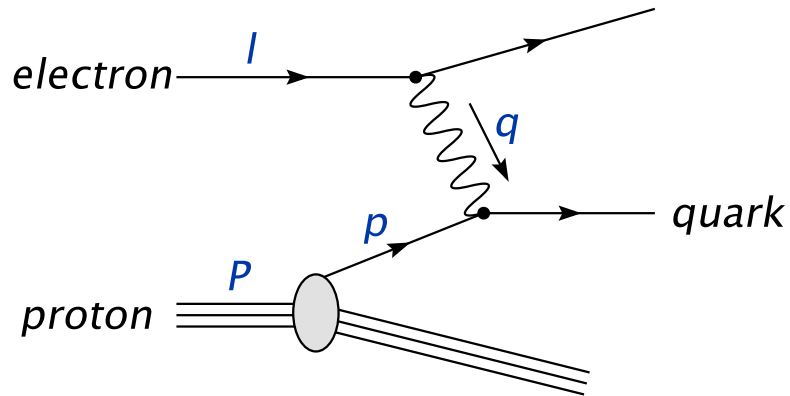
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- Deep-inelastic scattering (Bjorken limit: $Q^2 \rightarrow \infty$ and x fixed)
 Parton model (quasi-free point-like constituents, incoherence)

$$F_2(x, Q^2) \simeq F_2(x) = \sum_i e_i^2 x f_i(x)$$

- $x f_i(x)$ distribution for momentum fraction x of parton i

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2 / (2p \cdot q)$

- Structure function F_2^p (up to order $\mathcal{O}(1/Q^2)$)

$$x^{-1} F_2^p(x, Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi} C_{2,i} \left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2} \right) f_i^p(\xi, \mu^2)$$

- Coefficient functions

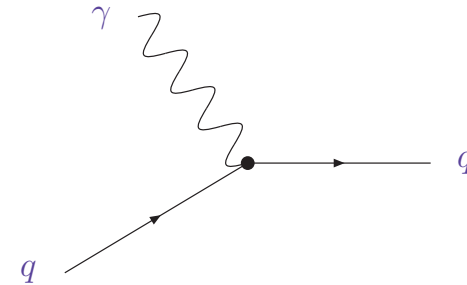
$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \alpha_s^4 c_{a,i}^{(4)} + \dots \right)$$

- current frontier in perturbation theory **N⁴LO** (work in progress)

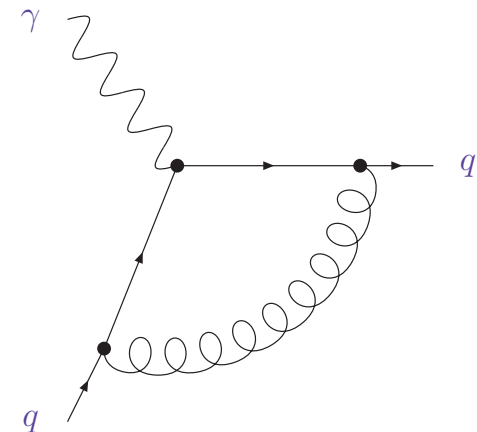
Radiative corrections in a nutshell

- Leading order
 - partonic structure function

$$\hat{F}_{2,q}^{(0)} = e_q^2 \delta(1-x)$$

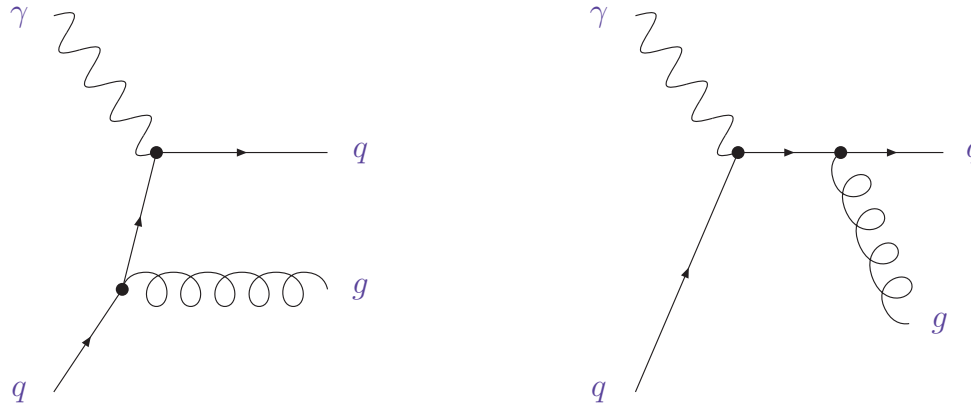


- Next-to-leading order
 - virtual correction
(infrared divergent; proportional to Born)
 - dimensional regularization $D = 4 - 2\epsilon$



$$\hat{F}_{2,q}^{(1),v} = e_q^2 C_F \frac{\alpha_s}{4\pi} \delta(1-x) \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \zeta_2 + \mathcal{O}(\epsilon) \right)$$

- Next-to-leading order



- add real and virtual corrections $\hat{F}_{2,q}^{(1)} = \hat{F}_{2,q}^{(1),r} + \hat{F}_{2,q}^{(1),v}$
- collinear divergence remains **splitting functions** $P_{qq}^{(0)}$

$$\begin{aligned} \hat{F}_{2,q}^{(1)} = & e_q^2 C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left(\frac{4}{1-x} - 2 - 2x + 3\delta(1-x) \right) \right. \\ & + 4 \frac{\ln(1-x)}{1-x} - 3 \frac{1}{1-x} - (9 + 4\zeta_2) \delta(1-x) \\ & - 2(1+x)(\ln(1-x) - \ln(x)) - 4 \frac{1}{1-x} \ln(x) + 6 + 4x \\ & \left. + \mathcal{O}(\epsilon) \right\} \end{aligned}$$

- Structure of NLO correction
 - absorb collinear divergence $P_{qq}^{(0)}$ in renormalized parton distributions

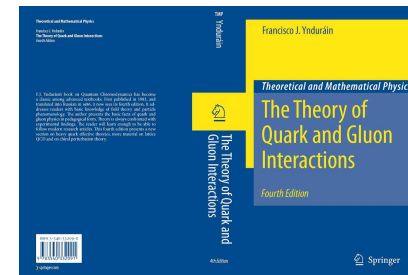
$$\hat{F}_{2,q}^{(1),\text{bare}} = e_q^2 \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} P_{qq}^{(0)}(x) + c_{2,q}^{(1)}(x) + \mathcal{O}(\epsilon) \right\}$$

$$f_q^{p,\text{ren}}(\mu_F^2) = f_q^{p,\text{bare}} - \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} P_{qq}^{(0)}(x) \left(\frac{\mu^2}{\mu_F^2} \right)^\epsilon \otimes f_q^{p,\text{bare}}$$

- partonic (physical) structure function at factorization scale μ_F

$$\hat{F}_{2,q} = e_q^2 \left(\delta(1-x) + \frac{\alpha_s}{4\pi} \left\{ c_{2,q}^{(1)}(x) - \ln \left(\frac{Q^2}{\mu_F^2} \right) P_{qq}^{(0)}(x) \right\} \right)$$

- Details in chapt. 4.8 of
The Theory of Quark and Gluon Interactions
 F.J. Yndurain



Wilson coefficients

- Recall QCD corrections to Wilson coefficients $c_{2,q}^{(1)}$ in $\hat{F}_{2,q}^{(1)}$ (cross section of hard parton scattering)

$$\begin{aligned} \hat{F}_{2,q}^{(1)} = & e_q^2 C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left(\frac{4}{1-x} - 2 - 2x + 3\delta(1-x) \right) \right. \\ & + 4 \frac{\ln(1-x)}{1-x} - 3 \frac{1}{1-x} - (9 + 4\zeta_2)\delta(1-x) \\ & \left. - 2(1+x)(\ln(1-x) - \ln(x)) - 4 \frac{1}{1-x} \ln(x) + 6 + 4x + \mathcal{O}(\epsilon) \right\} \end{aligned}$$

- large radiative corrections as $x \rightarrow 1$
- Mellin transform with moments N (integral transform $x \rightarrow N$)

$$f(N) = \int_0^1 dx x^{N-1} f(x)$$

- dictionary:

$[\ln(1-x)/(1-x)]_+$	$\rightarrow$	$\ln^2 N$
$[1/(1-x)]_+$	$\rightarrow$	$\ln N$
$\delta(1-x)$	$\rightarrow$	const_N

Threshold resummation

- Coefficient function in large x -limit have large logarithms at n^{th} -order

$$\alpha_s^n \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \ln^{2n}(N)$$

- Threshold resummation in Mellin space

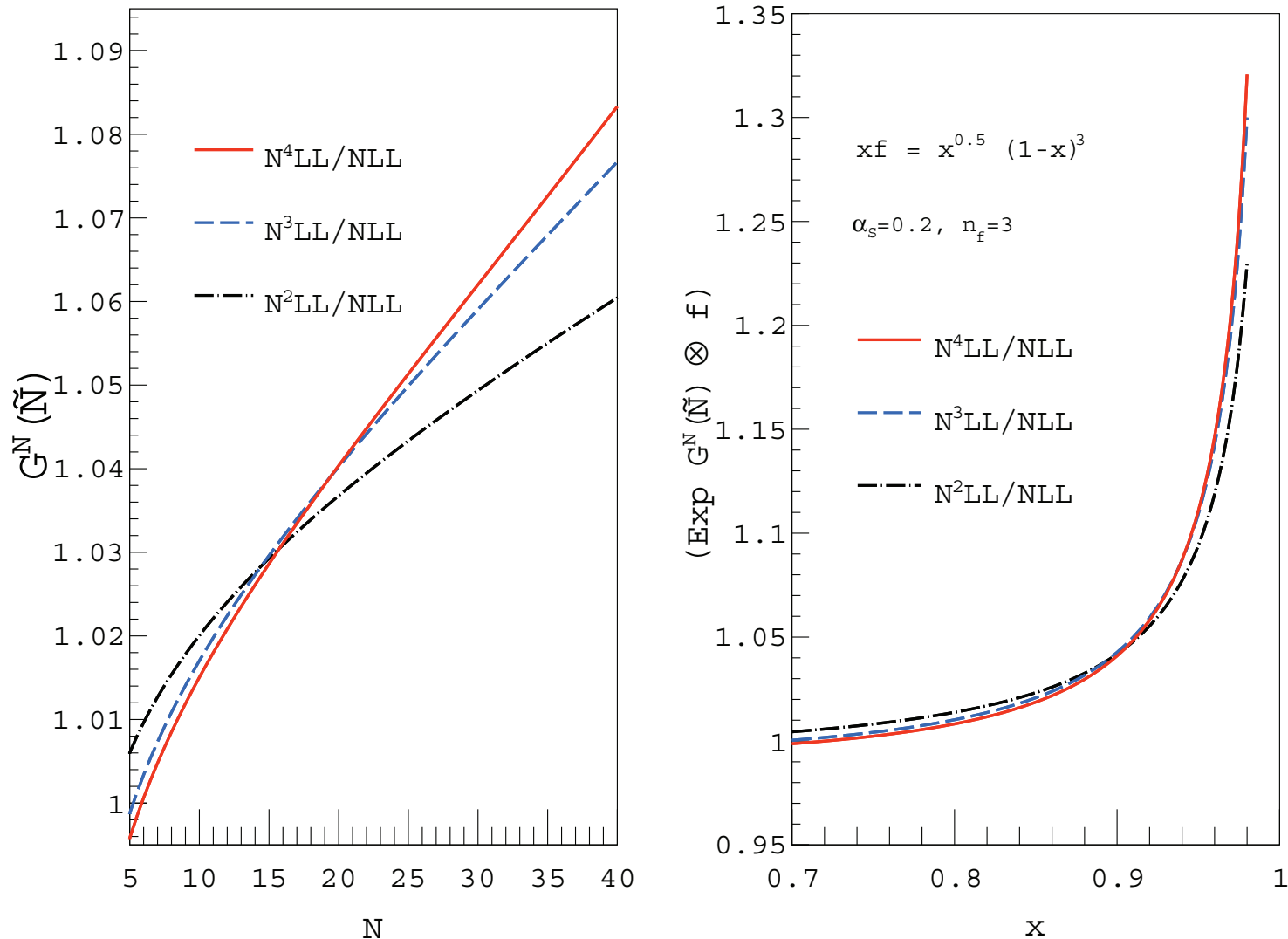
$$C^N = (1 + \alpha_s g_{01} + \alpha_s^2 g_{02} + \dots) \cdot \exp(G^N) + \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N^k LL accuracy

$$G^N = \ln(N)g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \dots$$

- $g_1(\lambda)$: LL Stermen '87; Appell, Mackenzie, Stermen '88
- $g_2(\lambda)$: NLL Catani Trenatdue '89
- $g_3(\lambda)$: NNLL or N^2 LL Vogt '00; Catani, Grazzini, de Florian, Nason '03
- $g_4(\lambda)$: N^3 LL S.M., Vermaseren, Vogt '05
- $g_5(\lambda)$: N^4 LL Das, S.M., Vogt '19
- Resummed G^N predicts fixed orders in perturbation theory
 - generating functional for towers of large logarithms

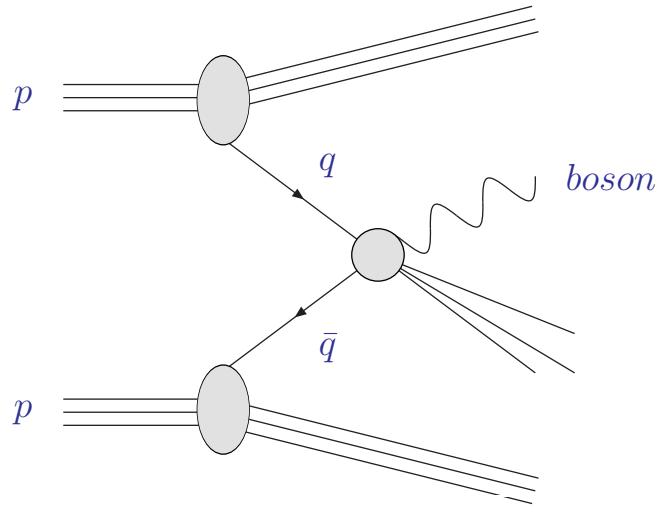
Numerical results for DIS



- Left: Resummed exponent G^N normalized to NLL for DIS plotted successively up to N^4LL for $\alpha_s = 0.2$ and $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution $xf = x^{0.5}(1-x)^3$ up to N^4LL

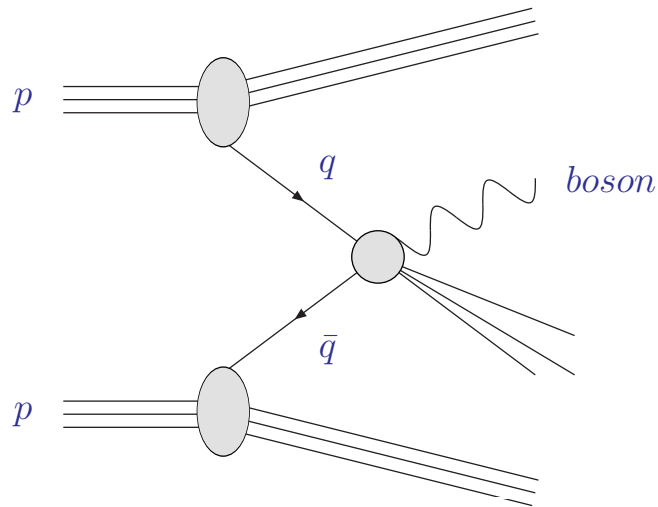
Drell-Yan process

Vector boson production

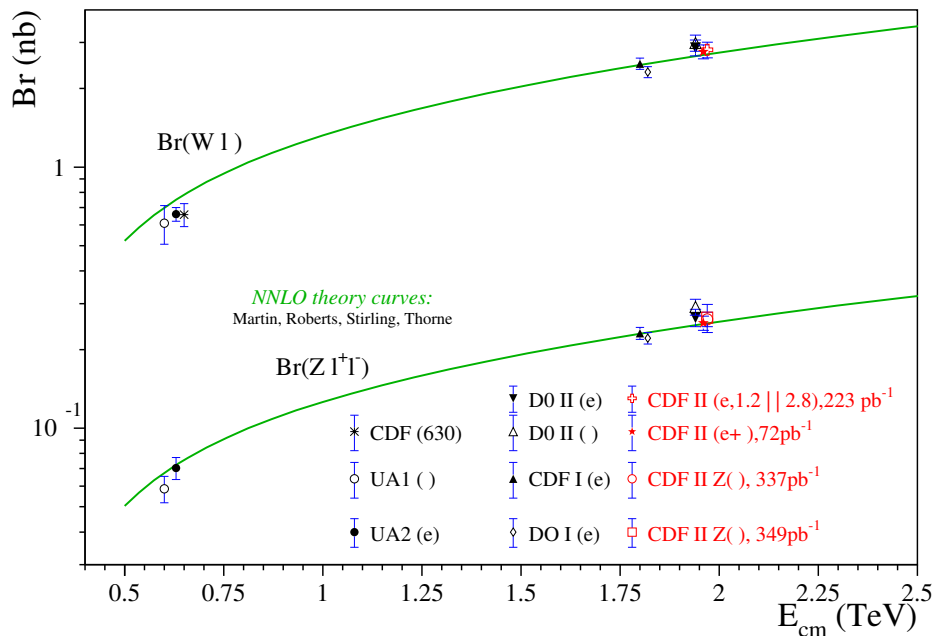


- Kinematical variables (inclusive)
 - energy (cms) $s = Q^2$ (time-like)
 - scaling variable $x = M_{W^\pm/Z}^2/s$

Vector boson production

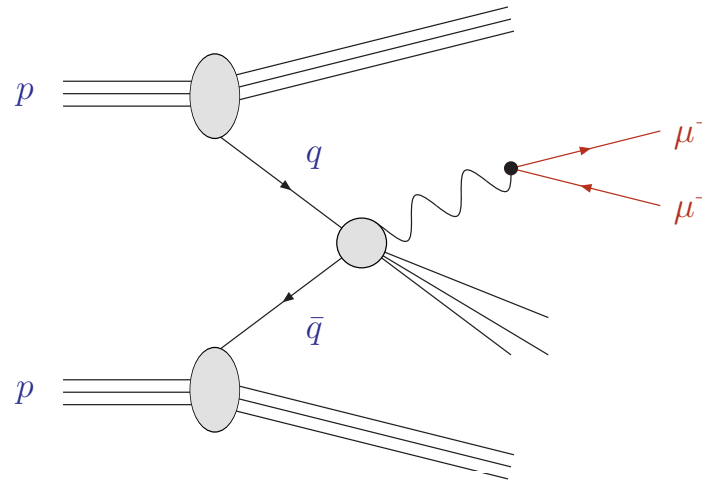


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- Early measurements of $W^\pm$ and Z cross sections at hadron colliders
CERN's SppS; Fermilab's Tevatron

QCD corrections to W/Z production



- Hadronic cross section $\sigma_{pp \rightarrow V}$ with $\tau = M_V^2/s$ and $V = \gamma^*/W^\pm/Z$
 - renormalization/factorization (hard) scale $\mu = \mathcal{O}(M_V)$

$$\sigma_{pp \rightarrow V} = \sum_{ij} \int_{\tau}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2}{x_2} f_i \left(\frac{x_1}{x_2}, \mu^2 \right) f_j(x_2, \mu^2) \hat{\sigma}_{ij \rightarrow V} \left(\frac{\tau}{x_1}, \frac{\mu^2}{M_V^2}, \alpha_s(\mu^2) \right)$$

- Partonic cross section $\hat{\sigma}_{ij \rightarrow V}$

$$\hat{\sigma}_{ij \rightarrow V} = \left[\underbrace{\hat{\sigma}_{ij \rightarrow V}^{(0)} + \alpha_s \hat{\sigma}_{ij \rightarrow V}^{(1)} + \alpha_s^2 \hat{\sigma}_{ij \rightarrow V}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^3 \hat{\sigma}_{ij \rightarrow V}^{(3)} + \dots \right]$$

NNLO: standard approximation

Kinematics (differential)

- Proton-proton scattering (two broad-band beams of incoming partons)
 - cms of parton-parton scattering boosted wrt incoming protons
- Final state variables (simple transformations under longitud. boosts)

$$p^\mu = (E, p_x, p_y, p_z) = (m_t \cosh y, p_t \sin \phi, p_t \cos \phi, m_t \sinh y)$$

- rapidity $y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right)$
- transverse momentum p_t and mass $m_t = \sqrt{p_t^2 + m^2}$
- azimuthal angle ϕ

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- transverse momentum p_t and mass $m_t = \sqrt{p_t^2 + m^2}$
- azimuthal angle ϕ
- Differences in rapidity Δy and azimuthal angle $\Delta \phi$ invariant under boosts
- In practice (for $E \gg m_p$)
 - pseudo-rapidity $\eta = -\ln \tan \left(\frac{\theta}{2} \right)$ with angle from beam axis

Differential distributions (I)

- Cross section for $\hat{\sigma}_{q\bar{q} \rightarrow e^+e^-}$

- Born cross section

$$\hat{\sigma}_{q\bar{q} \rightarrow e^+e^-} = \frac{4\pi\alpha^2}{3s} \frac{e_q^2}{N_c} = \sigma^{(0)} \frac{e_q^2}{N_c}$$

- Born result for invariant mass distribution $\frac{d\hat{\sigma}}{dM^2}$

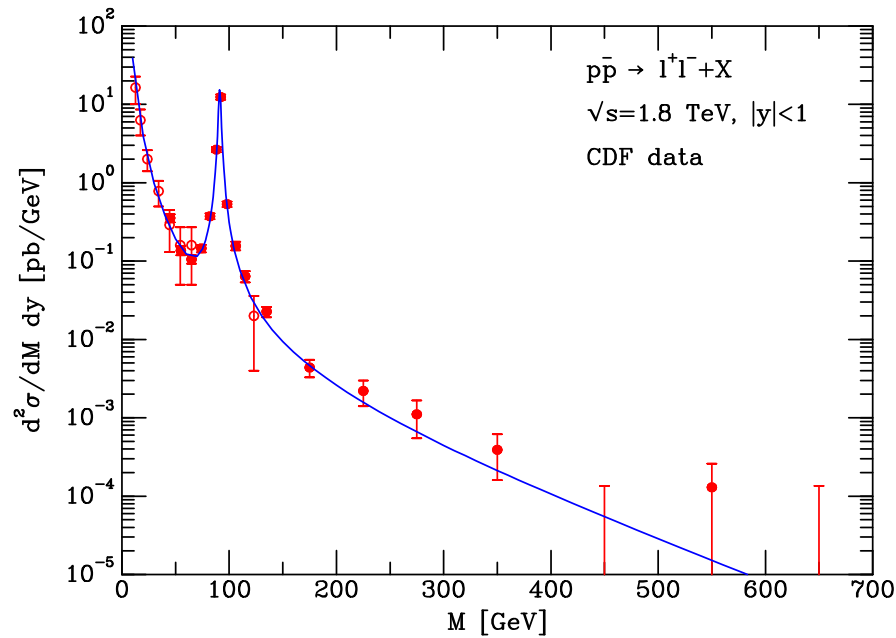
- use of $s = M_V^2$ implies $\delta(s - M^2)$

$$M^2 \frac{d\hat{\sigma}}{dM^2} = \sigma^{(0)} \frac{e_q^2}{N_c} \delta(s - M^2)$$

- Hadronic cross section from convolution with parton distributions
 - Born result

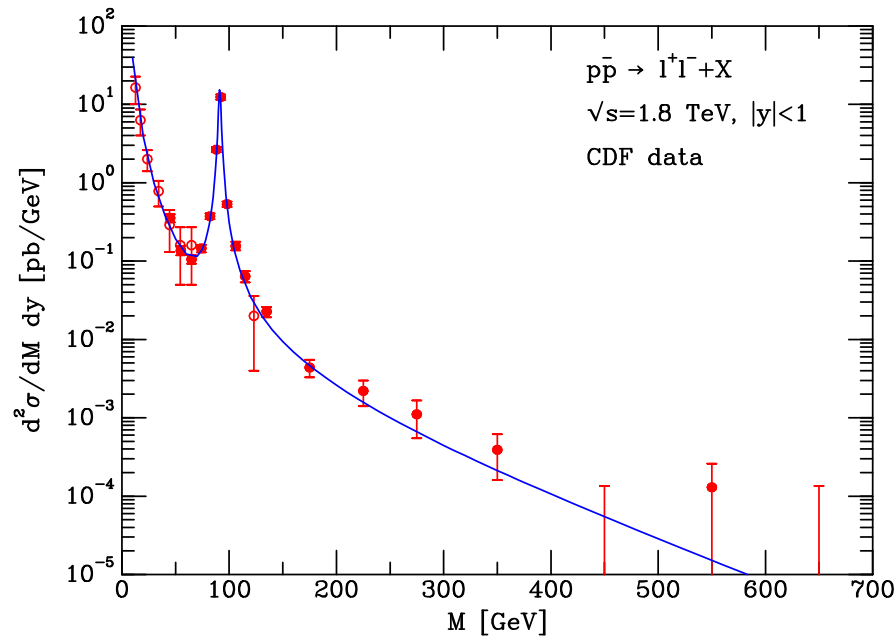
$$M^4 \frac{d\sigma}{dM^2} = \sigma^{(0)} \frac{1}{N_c} \frac{M^2}{s} \times \int_0^1 dx_1 dx_2 \delta\left(x_1 x_2 - \frac{M^2}{s}\right) \sum_q e_q^2 \{f_q(x_1) f_{\bar{q}}(x_2) + f_{\bar{q}}(x_1) f_q(x_2)\}$$

Differential distributions (II)



- Invariant mass distribution $\frac{d\sigma}{dM^2}$ of lepton pair for Z -production in $p\bar{p}$ -collisions
 - CDF data at $\sqrt{s} = 1.8 \text{ TeV}$ and NLO QCD prediction

Differential distributions (II)



- Invariant mass distribution $\frac{d\sigma}{dM^2}$ of lepton pair for Z -production in $p\bar{p}$ -collisions
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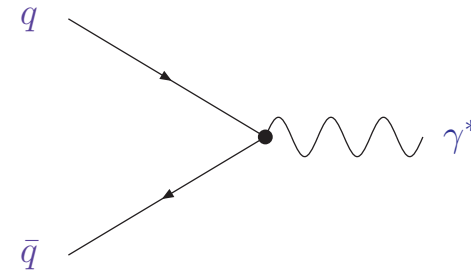
- Double-differential cross section $\frac{d\sigma}{dM^2 dy}$ local in PDFs
 - $y = \frac{1}{2} \ln \left(\frac{x_1}{x_2} \right)$ lepton-pair rapidity

Radiative corrections in a nutshell

- Leading order

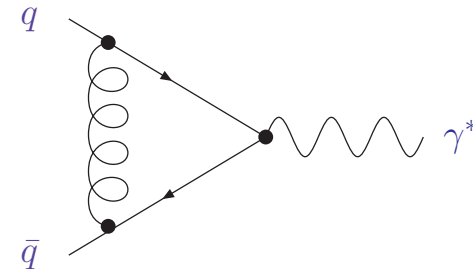
- partonic cross section $x = \tau/x_1$

$$\hat{\sigma}_{q\bar{q} \rightarrow V}^{(0)} = \delta(1-x)$$



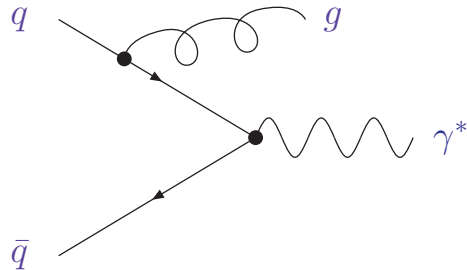
- Next-to-leading order

- virtual correction
(infrared divergent; proportional to Born)
- dimensional regularization $D = 4 - 2\epsilon$



$$\hat{\sigma}_{q\bar{q} \rightarrow V}^{(1),v} = C_F \frac{\alpha_s}{4\pi} \delta(1-x) \left(\frac{\mu^2}{M_V^2} \right)^\epsilon \left(-\frac{4}{\epsilon^2} - \frac{6}{\epsilon} - 16 + 2\zeta_2 + \mathcal{O}(\epsilon) \right)$$

- Next-to-leading order



- add real and virtual corrections $\hat{\sigma}_{q\bar{q} \rightarrow V}^{(1)} = \hat{\sigma}_{q\bar{q} \rightarrow V}^{(1),r} + \hat{\sigma}_{q\bar{q} \rightarrow V}^{(1),v}$
- collinear divergence remains in **splitting function** $P_{qq}^{(0)}$

$$\begin{aligned} \hat{\sigma}_{q\bar{q} \rightarrow V}^{(1)} = & \frac{\alpha_s}{4\pi} C_F \left(\frac{\mu^2}{M_V^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left(\frac{8}{1-x} - 4 - 4x + 6\delta(1-x) \right) \right. \\ & + \left(16 \frac{\ln(1-x)}{1-x} + (-16 + 8\zeta_2) \delta(1-x) \right. \\ & \left. \left. - 4 \frac{1+x^2}{1-x} \ln(x) + 8 \frac{1+x^2}{1-x} \ln(1-x) \right) \right. \\ & \left. + \mathcal{O}(\epsilon) \right\} \end{aligned}$$

- Structure of NLO correction
 - absorb collinear divergence $P_{qq}^{(0)}$ in renormalized parton distributions

$$\hat{\sigma}_{q\bar{q}\rightarrow V}^{(1),\text{bare}} = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{M_V^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} 2 P_{qq}^{(0)}(x) + \hat{\sigma}_{q\bar{q}\rightarrow V}^{(1)}(x) + \mathcal{O}(\epsilon) \right\}$$

$$f_q^{p,\text{ren}} = f_q^{p,\text{bare}} - \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} P_{qq}^{(0)}(x) \left(\frac{\mu^2}{\mu_F^2} \right)^\epsilon \otimes f_q^{p,\text{bare}}$$

- partonic (physical) structure function at factorization scale μ_F

$$\hat{\sigma}_{q\bar{q}\rightarrow V} = \delta(1-x) + \frac{\alpha_s}{4\pi} \left\{ \hat{\sigma}_{q\bar{q}\rightarrow V}^{(1)}(x) - \ln \left(\frac{M_V^2}{\mu_F^2} \right) 2 P_{qq}^{(0)}(x) \right\}$$

Coefficient functions

- QCD corrections to coefficient function of hard scattering in $\hat{\sigma}_{q\bar{q}\rightarrow V}^{(1)}$

$$\begin{aligned}\hat{\sigma}_{q\bar{q}\rightarrow V}^{(1)} = & \frac{\alpha_s}{4\pi} C_F \left(\frac{\mu^2}{M_V^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left(\frac{8}{1-x} - 4 - 4x + 6\delta(1-x) \right) \right. \\ & + \left(16 \frac{\ln(1-x)}{1-x} + (-16 + 8\zeta_2) \delta(1-x) \right. \\ & \left. \left. - 4 \frac{1+x^2}{1-x} \ln(x) + 8 \frac{1+x^2}{1-x} \ln(1-x) \right) \right. \\ & \left. + \mathcal{O}(\epsilon) \right\}\end{aligned}$$

- large radiative corrections as $x \rightarrow 1$

Threshold resummation

- Improved predictions for Drell-Yan process near production threshold by resumming large logarithms
- Enhanced cross-section accuracy by accounting for soft and collinear radiation effects

Summary (part III)

Perturbative QCD at work

- Factorization and evolution
 - scattering with initial state hadrons requires collinear factorization
 - separation of long and short distance physics
 - factorization induces evolution equations via renormalization group
 - parton distribution function

Inclusive DIS

- NLO QCD corrections
 - illustration of factorization, infrared safety and evolution for $ep \rightarrow X$
 - intuitive picture for parton evolution and proton structure

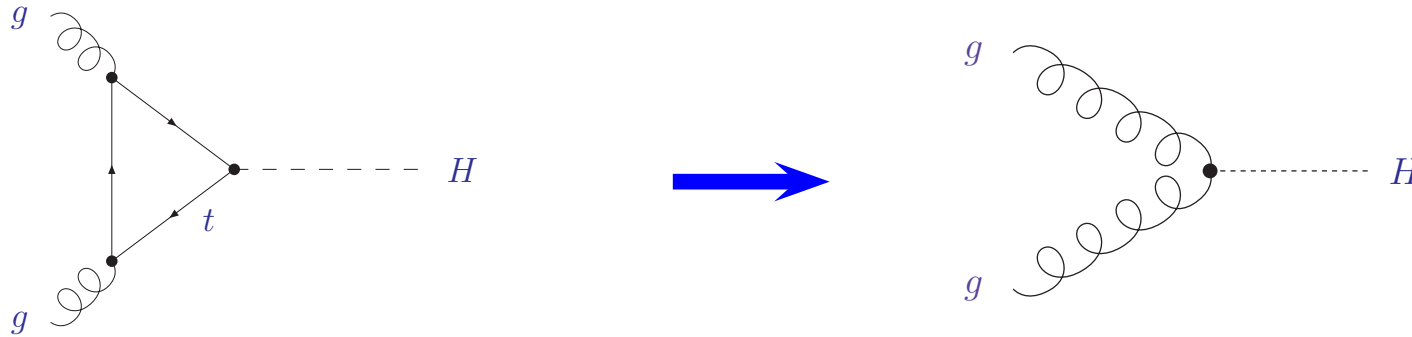
$W^\pm/Z$ -boson production

- $W^\pm/Z$ -boson production at hadron colliders
- NLO QCD corrections
 - illustration of factorization, infrared safety and evolution for $q\bar{q} \rightarrow V$

Back-up

Higgs boson production in gg -fusion

Effective theory

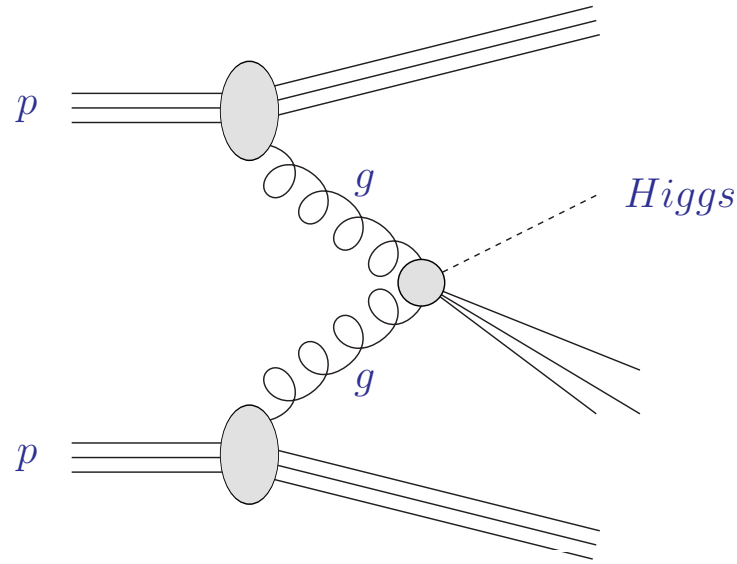


- Integration of top-quark loop (finite result)
 - decay width $H \rightarrow gg$ ($m_q = 0$ for light quarks, m_t heavy)

$$\Gamma_{H \rightarrow gg} = \frac{G_\mu m_H^3}{64 \sqrt{2} \pi^3} \alpha_s^2 f\left(\frac{m_H^2}{4m_t^2}\right)$$

- Effective theory in limit $m_t \rightarrow \infty$; Lagrangian $\mathcal{L} = -\frac{1}{4} \frac{H}{v} C_H G^{\mu\nu a} G_{\mu\nu}^a$
 - operator $H G^{\mu\nu a} G_{\mu\nu}^a$ relates to stress-energy tensor
 - additional renormalization proportional to QCD β -function required
Kluberg-Stern, Zuber '75; Collins, Duncan, Joglekar '77

QCD corrections to ggF



- Hadronic cross section $\sigma_{pp \rightarrow H}$ with $\tau = m_H^2/S$
 - renormalization/factorization (hard) scale $\mu = \mathcal{O}(m_H)$

$$\sigma_{pp \rightarrow H} = \sum_{ij} \int_{\tau}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2}{x_2} f_i \left(\frac{x_1}{x_2}, \mu^2 \right) f_j (x_2, \mu^2) \hat{\sigma}_{ij \rightarrow H} \left(\frac{\tau}{x_1}, \frac{\mu^2}{m_H^2}, \alpha_s(\mu^2) \right)$$

- Partonic cross section $\hat{\sigma}_{ij \rightarrow H}$

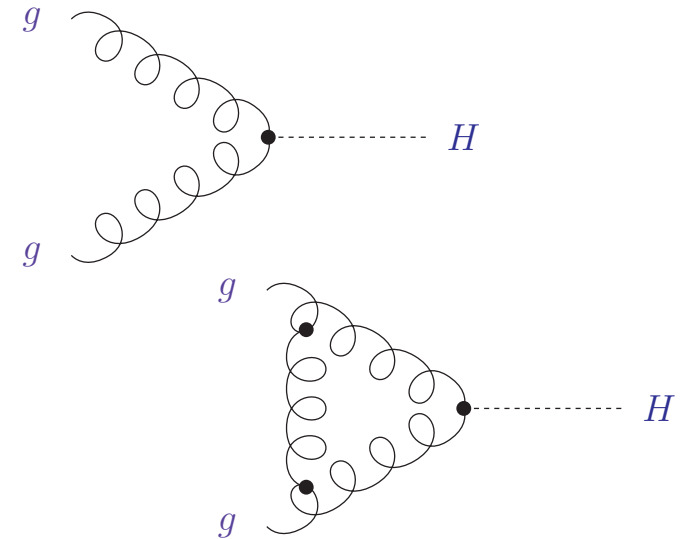
$$\hat{\sigma}_{ij \rightarrow H} = \underbrace{\alpha_s^2 \left[\hat{\sigma}_{ij \rightarrow H}^{(0)} + \alpha_s \hat{\sigma}_{ij \rightarrow H}^{(1)} + \alpha_s^2 \hat{\sigma}_{ij \rightarrow H}^{(2)} \right]}_{\text{NNLO: standard approximation}} + \alpha_s^3 \hat{\sigma}_{ij \rightarrow H}^{(3)} + \dots$$

NNLO: standard approximation

Radiative corrections in a nutshell

- Leading order
 - partonic cross section $x = \tau/x_1$

$$\hat{\sigma}_{gg \rightarrow H}^{(0)} = \delta(1-x)$$
- Next-to-leading order
 - virtual correction (time-like kinematics)
(infrared divergent; proportional to Born)
 - dimensional regularization $D = 4 - 2\epsilon$

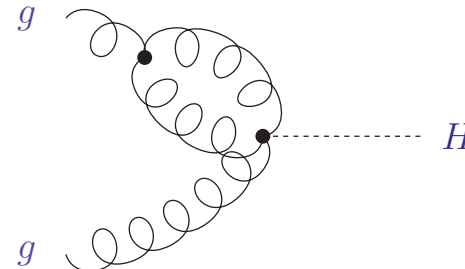


$$\hat{\sigma}_{gg \rightarrow H}^{(1),v} = C_A \frac{\alpha_s}{4\pi} \delta(1-x) \left(\frac{\mu^2}{m_H^2} \right)^\epsilon \left(-\frac{2}{\epsilon^2} + 7\zeta_2 + \mathcal{O}(\epsilon) \right)$$

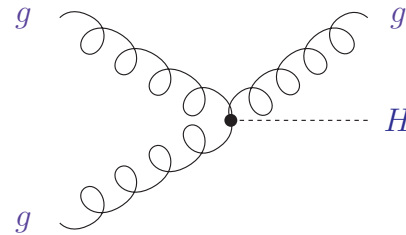
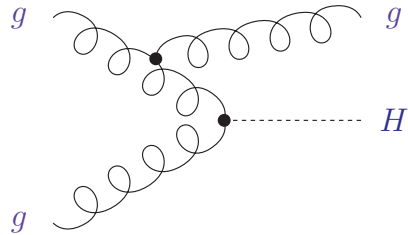
- additional contribution from renormalization of effective operator

$$\alpha_s^{\text{bare}} = \alpha_s^{\text{ren}} \left\{ 1 - \frac{\beta_0}{\epsilon} \frac{\alpha_s^{\text{ren}}}{4\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

- massless tadpoles vanish
in dimensional regularization



- Next-to-leading order



- add real and virtual corrections $\hat{\sigma}_{gg \rightarrow H}^{(1)} = \hat{\sigma}_{gg \rightarrow H}^{(1),r} + \hat{\sigma}_{gg \rightarrow H}^{(1),v}$
- collinear divergence remains in **splitting function** $P_{gg}^{(0)}$

$$\hat{\sigma}_{gg \rightarrow H}^{(1)} = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{m_H^2} \right)^\epsilon \left\{ \right.$$

$$\frac{1}{\epsilon} C_A \left(\frac{8}{1-x} + \frac{8}{x} - 8(2-x+x^2) + \frac{22}{3} \delta(1-x) \right) - \frac{1}{\epsilon} n_f \frac{4}{3} \delta(1-x)$$

$$+ C_A \left(16 \frac{\ln(1-x)}{1-x} + \left(\frac{22}{3} + 8\zeta_2 \right) \delta(1-x) - 16x(2-x+x^2) \ln(1-x) \right.$$

$$\left. - 8 \frac{(1-x+x^2)^2}{1-x} \ln(x) - \frac{22}{3} (1-x)^3 \right)$$

$$+ \mathcal{O}(\epsilon) \left. \right\}$$

- Structure of NLO correction
 - absorb collinear divergence $P_{gg}^{(0)}$ in renormalized parton distributions

$$\hat{\sigma}_{gg \rightarrow H}^{(1), \text{bare}} = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{m_H^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} 2 P_{gg}^{(0)}(x) + \hat{\sigma}_{gg \rightarrow H}^{(1)}(x) + \mathcal{O}(\epsilon) \right\}$$

$$g^{\text{ren}}(\mu_F^2) = g^{\text{bare}} - \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} P_{gg}^{(0)}(x) \left(\frac{\mu^2}{\mu_F^2} \right)^\epsilon \otimes g^{\text{bare}}$$

- partonic (physical) structure function at factorization scale μ_F

$$\hat{\sigma}_{gg \rightarrow H} = \delta(1-x) + \frac{\alpha_s}{4\pi} \left\{ \hat{\sigma}_{gg \rightarrow H}^{(1)}(x) - \ln \left(\frac{m_H^2}{\mu_F^2} \right) 2 P_{gg}^{(0)}(x) \right\}$$