Quantum Chromodynamics

lecture I

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IMSc Spring School on High Energy Physics Chennai, Feb 24, 2025

Plan

- *Introduction to QCD*
  Monday, February 24, 2025

- QCD at work: infrared safety and jets
  Tuesday, February 25, 2025

- QCD at work: factorization and evolution
  Wednesday, February 26, 2025

- Deep structure of proton
  Thursday, February 27, 2025

Quantum Chromodynamics

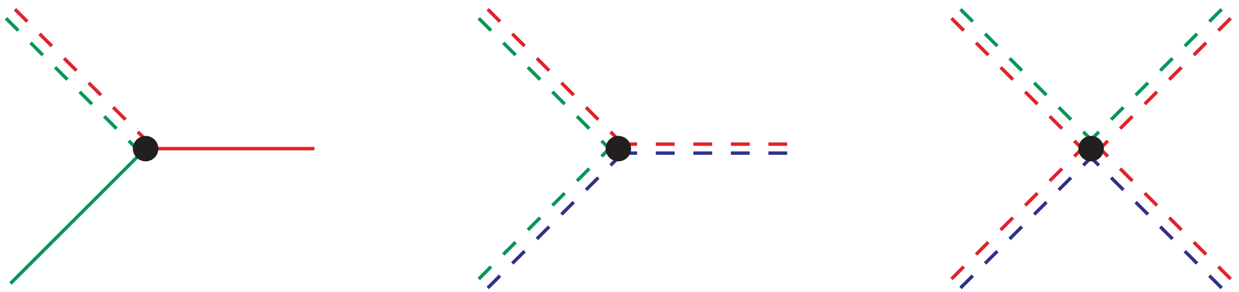
- Fundamental forces: quantum fields with gauge symmetries
- Strong interaction: color-$SU(3)$
  - quarks (antiquarks): 6 spin-1/2 flavors in 3 colors

$$\begin{bmatrix} u_{\rm B} \\ u_{\rm G} \\ u_{\rm R} \end{bmatrix} \begin{bmatrix} d_{\rm B} \\ d_{\rm G} \\ d_{\rm R} \end{bmatrix} , \quad \begin{bmatrix} s_{\rm B} \\ s_{\rm G} \\ s_{\rm R} \end{bmatrix} \begin{bmatrix} c_{\rm B} \\ c_{\rm G} \\ c_{\rm R} \end{bmatrix} , \quad \begin{bmatrix} b_{\rm B} \\ b_{\rm G} \\ b_{\rm R} \end{bmatrix} \begin{bmatrix} t_{\rm B} \\ t_{\rm G} \\ t_{\rm R} \end{bmatrix}$$

  - gluons: 8 spin-1 color-anticolor-combinations

$$g_{\rm B\bar{G}} \,,\; g_{\rm R\bar{B}} \,,\; g_{\rm G\bar{R}} \,,\; \cdots \,,\; g_{\rm B\bar{B}-G\bar{G}} \,,\; g_{\rm B\bar{B}+G\bar{G}-2\,R\bar{R}}$$

- Interactions: Feynman diagrams



QCD Lagrangian

- Classical part of QCD Lagrangian

$$\mathcal{L}_{\rm cl} \quad = \quad -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \sum_{\rm flavors} \bar{\psi}_i \left({\rm i} \not{D} - m_q \right)_{ij} \psi_j$$

- Matter fields $\psi_i, \bar{\psi}_j$ with $i, j = 1, \dots, 3$ (fundamental rep.)
  - covariant derivative $D_{\mu,ij} = \partial_\mu \delta_{ij} + {\rm i} g_s \left(t_a \right)_{ij} A_\mu^a$
- Field strength tensor $F_{\mu\nu}^a$ with $a = 1, \dots, 8$ (adjoint rep.)
  - covariant derivative $D_{\mu,ab} = \partial_\mu \delta_{ab} - g_s f_{abc} A_\mu^c$
  - $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc} A_\mu^b A_\nu^c$
- Formal parameters of the theory (no observables)
  - strong coupling $\alpha_s = g_s^2 / (4\pi)$
  - quark masses m_q

Quantization

- Gauge fixing (Feynman gauge $\lambda = 1$) $\mathcal{L}_{\rm gauge-fix} \;=\; -\frac{1}{2\lambda} \left(\partial^\mu A_\mu^a \right)^2$
- Ghosts (Grassmann fields η) $\mathcal{L}_{\rm ghost} \;=\; \partial_\mu \eta^{a\dagger} \left(D_{ab}^\mu \eta^b \right)$
  (removal of unphysical degrees of freedom for gauge fields) Fadeev, Popov

From Lagrangian to Feynman rules

- Consider action S

$$S = \mathrm{i} \int d^4x \, (\mathcal{L}_{\rm cl} + \mathcal{L}_{\rm gauge-fix} + \mathcal{L}_{\rm ghost}) = S_{\rm free} + S_{\rm int}$$

- Decompose action into free $S_{\rm free}$ and interacting part $S_{\rm int}$
  - $S_{\rm free}$ contains bi-linear terms in fields
  - $S_{\rm int}$ contains interactions
- Derivation of Feynman rules
  - inverse propagators from $S_{\rm free}$
  - interacting parts from $S_{\rm int}$ (in perturbative expansion)

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Examples (I)

- Fermion propagator in QCD from $\bar{\psi}_i \delta_{ij} \, (\mathrm{i}\not{\partial} - m_q) \, \psi_j$
  - substitution $\partial_\mu = -\mathrm{i} p_\mu$ (Fourier transformation)
- Inverse propagator (momentum space) $\Gamma_{ij}^{\bar{\psi}\psi}(p) = -\mathrm{i} \, \delta_{ij} \, (\not{p} - m_q)$
- Check: quark propagator $\Delta_{ij}(p) = +\mathrm{i} \, \delta_{ij} \, \dfrac{1}{\not{p} - m_q + \mathrm{i}0}$
  - causality in Minkowski space: prescription $+\mathrm{i}0$

Examples (II)

- Gluon propagator in QCD from bi-linear terms in $F^a_{\mu\nu} F_a^{\mu\nu}$ and $\mathcal{L}_{\text{gauge-fix}}$
  - recall $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$
  - recall $\mathcal{L}_{\text{gauge-fix}} = -\frac{1}{2\lambda} \left(\partial^\mu A^a_\mu\right)^2$
- Inverse propagator (momentum space)

$$\Gamma^{AA}_{ab;\mu\nu}(p) = +\mathrm{i}\,\delta_{ab} \left[p^2 g_{\mu\nu} - \left(1 - \frac{1}{\lambda}\right) p_\mu p_\nu\right]$$

- Gluon propagator $\Delta^{ab;\mu\nu}(p) = +\mathrm{i}\,\delta_{ab} \left[\frac{-g_{\mu\nu}}{p^2} + (1-\lambda)\frac{p_\mu p_\nu}{p^4}\right]$
  - Check: $\Gamma^{AA}_{ac;\mu\rho}(p)\,\Delta^{cb;\rho\nu}(p) = \delta_a^{\ b}\, g_\mu^{\ \nu}$

Examples (II)

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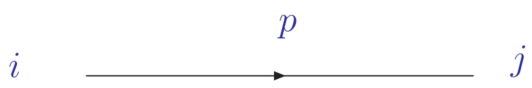
  - Check: $\Gamma^{AA}_{ac;\mu\rho}(p)\, \Delta^{cb;\rho\nu}(p) = \delta_a^{\ b}\, g_\mu^{\ \nu}$

Examples (III)

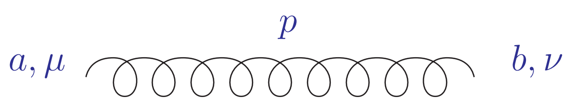
- Interactions derived from S_{int}
  - fermion-gluon interaction from $\bar{\psi}_i \mathrm{i} \not{A}_{ij} \psi_j \longrightarrow -\mathrm{i} t^a_{ij} \gamma_\mu$
- General rule
  - replacement of all ∂_μ by momenta p_μ
    (tedious for 3- and 4-gluon interactions)

Feynman rules (I)

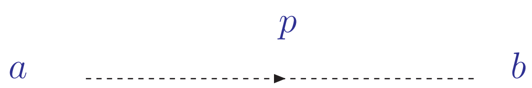
- Propagators
  - fermions, gluons, ghosts
  - covariant gauge



$$\delta^{ij} \, \frac{\mathrm{i}}{\not{p} - m}$$



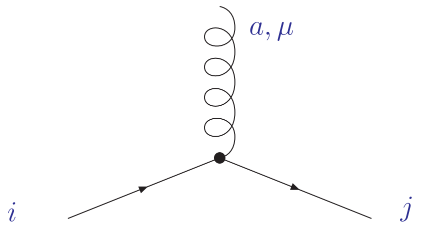
$$\delta^{ab} \, \mathrm{i} \left(\frac{-g^{\mu\nu}}{p^2} + (1 - \lambda) \frac{p^\mu p^\nu}{(p^2)^2} \right)$$



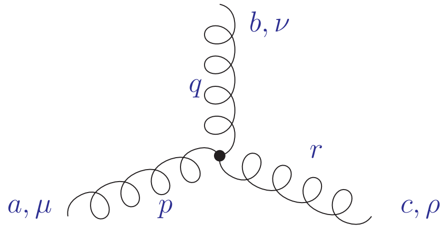
$$\delta^{ab} \, \frac{\mathrm{i}}{p^2}$$

Feynman rules (II)

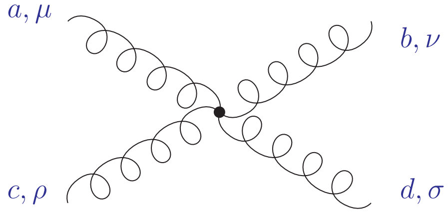
- Vertices



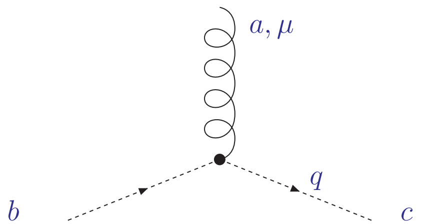
$$-\mathrm{i}\, g\ (t^a)_{ji}\ \gamma^\mu$$



$$-g\, f^{abc}\ \left((p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\mu\rho}\right)$$



$$-\mathrm{i}\, g^2\, f^{xac} f^{xbd}\ \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right)$$
$$-\mathrm{i}\, g^2\, f^{xad} f^{xbc}\ \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right)$$
$$-\mathrm{i}\, g^2\, f^{xab} f^{xcd}\ \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right)$$



$$g\, f^{abc}\, q^\mu$$

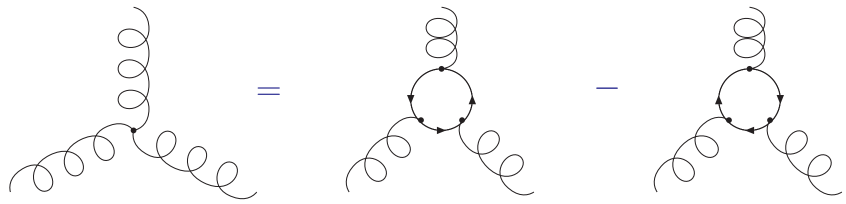
Color algebra

- $SU(N)$-generators t^a from fundamental representation

$$\mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}$$

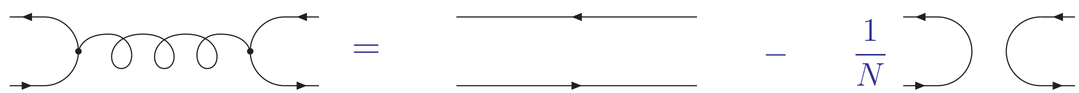
- $SU(N)$-generators f^{abc} of adjoint representation

$$f^{abc} = \mathrm{i}\,\mathrm{Tr}\left(\left[t^a, t^b\right] t^c\right)$$



- Fierz identity

$$\left(t^a\right)_{i_1}^{\ j_1} \left(t^b\right)_{i_2}^{\ j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}$$



Useful relations

- Color constant (quadratic Casimir)
  - quarks $\left(t^a t^a\right)_{ij} = C_F \delta_{ij} = \dfrac{N^2-1}{2N}\delta_{ij} = \dfrac{4}{3}\delta_{ij}$
  - gluons $f^{acd} f^{bcd} = C_A \delta_{ab} = N\delta_{ab} = 3\delta_{ab}$

Renormalization (I)

Physics picture

- Parameters of Lagrangian in quantum field theory have no unique physical interpretation
- Generic quantity R depends on
  - hard scale Q, mass m_q
  - in perturbative study on coupling constant α_s
- Radiative corrections
  - resolve quantum fluctuations at given resolution length $a \sim 1/\mu$
  - induce dependence of R on scale μ

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  - induce dependence of R on scale μ
- Renormalization “group” governed by QCD describes changes R with respect to μ (differential equation of first order)

$$\left\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \gamma_m(\alpha_s)\, m_q \frac{\partial}{\partial m_q} \right\} R\left(\frac{Q^2}{\mu^2}, \alpha_s, \frac{m_q^2}{Q^2}\right) = 0$$

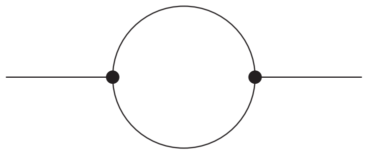
  - partial derivatives $\beta(\alpha_s) = \frac{\partial}{\partial \mu^2}\alpha_s$ and $\gamma_m(\alpha_s)\, m_q = \frac{\partial}{\partial \mu^2} m_q$
  - solution of differential equation requires initial conditions $\longrightarrow$ definition of renormalization scheme

Renormalization (II)

Technicalities

- Radiative corrections require integration over loop momenta
  - loop integrals can diverge in ultraviolet $l \to \infty$
  - power counting reveals divergence in ultraviolet
- Example: self-energy in scalar field theory (off-shell momentum $q^2 \neq 0$)

$$\int d^4l \, \frac{1}{l^2(l-q)^2}$$



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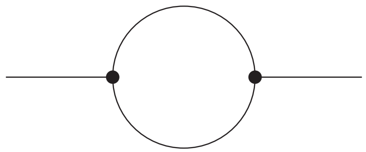
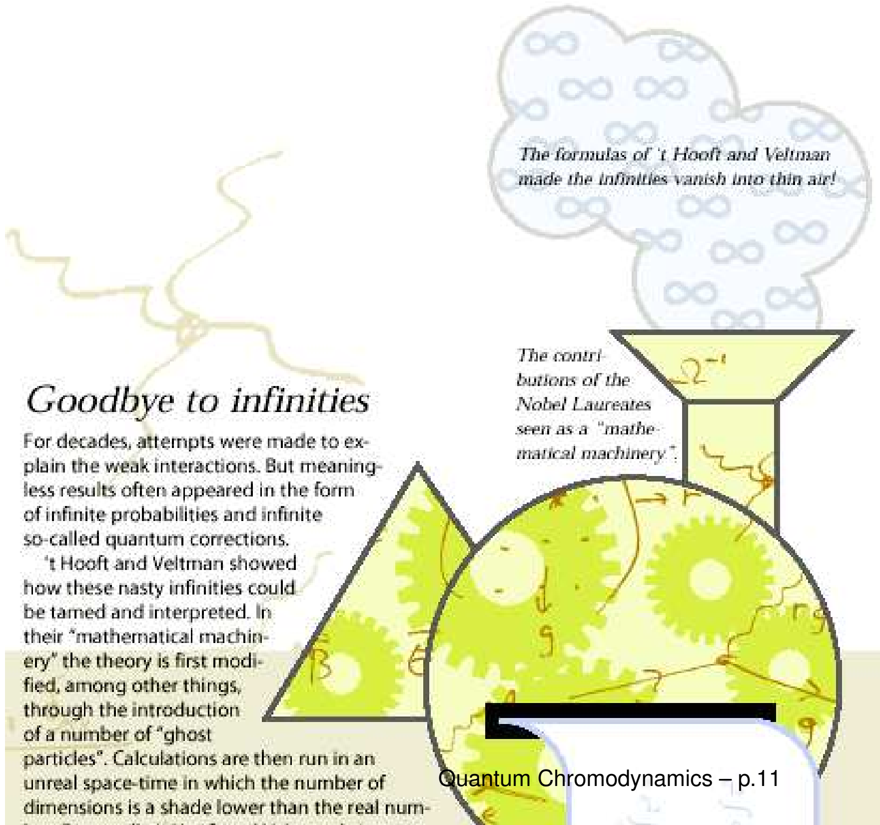


Nobel prize 1999



Goodbye to infinities

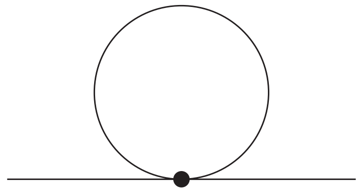
For decades, attempts were made to explain the weak interactions. But meaningless results often appeared in the form of infinite probabilities and infinite so-called quantum corrections.

't Hooft and Veltman showed how these nasty infinities could be tamed and interpreted. In their "mathematical machinery" the theory is first modified, among other things, through the introduction of a number of "ghost particles". Calculations are then run in an unreal space-time in which the number of dimensions is a shade lower than the real num-

Regularization

- Example for UV divergent loop integral (Euclidean region):

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 + m_q^2}$$



- Various regularization methods
  - cut-off $l \leq \Lambda_{\mathrm{cut-off}}$; Pauli-Villars $\frac{1}{l^2} \to \frac{1}{l^2 - M^2}$; lattice; ...

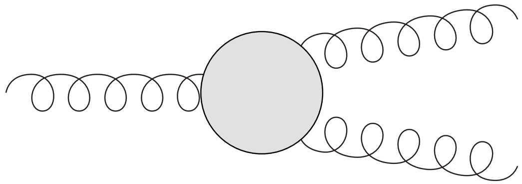
Dimensional regularization

- Lorentz invariance and $SU(N)$ gauge invariance manifest
- Anlytical continuation in space-time dimension $D = 4 - 2\epsilon$
  - loop integral $\int \frac{d^4 l}{(2\pi)^4} \to \int \frac{d^D l}{(2\pi)^D}$
  - Lorentz index $\mu \in \{0, 1, 2, 3\} \to \{0, 1, \ldots, D\}$
  - Lorentz vector $p^\mu \in \left(p^0, p^1, p^2, p^3\right) \to \left(p^0, p^1, \ldots, p^{D-1}\right)$
  - metric $g^{\mu\nu} g_{\mu\nu} = g^\mu_\mu = D$
  - Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^\mu \gamma^\nu \gamma_\mu = (2 - D)\gamma^\nu$

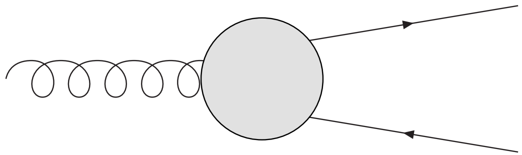
Renormalization (III)

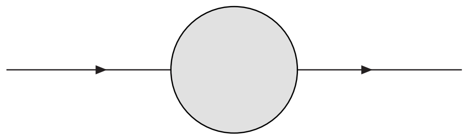
In a nut-shell

- compute vertex corrections
  (one-particle irreducible diagrams)
  - three-gluon vertex
    $V_{A_\mu A_\nu A_\rho} = g_s A_\mu A_\nu A_\rho$

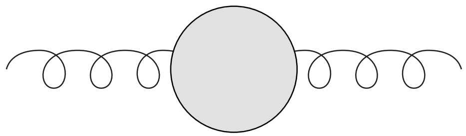


  - gluon-quark vertex
    $V_{\psi\bar{\psi}A_\mu} = g_s A_\mu \bar{\psi}\psi$





- compute self-energy corrections



- Redefinition of fields, vertices and parameters in Lagrangian
  - vertices $V^{\mathrm{b}}_{A_\mu A_\nu A_\rho} = Z_1 V^{\mathrm{r}}_{A_\mu A_\nu A_\rho}$ and $V^{\mathrm{b}}_{\psi\bar{\psi}A_\mu} = Z_{1F} V^{\mathrm{r}}_{\psi\bar{\psi}A_\mu}$
  - fields $\psi^{\mathrm{b}} = (Z_2)^{1/2}\,\psi^{\mathrm{r}}$ and $A^{\mathrm{b}}_\mu = (Z_3)^{1/2}\,A^{\mathrm{r}}_\mu$
  - parameters coupling constant $g_s^{\mathrm{b}} = Z_g g_s^{\mathrm{r}}$ and mass $m^{\mathrm{b}} = Z_m m^{\mathrm{r}}$

Renormalization (IV)

Gauge invariance

- Renormalization of vertex corrections imply

$$Z_1 \;=\; Z_g\,(Z_3)^{3/2}\;, \qquad\qquad Z_{1F} \;=\; Z_g\,(Z_3)^{1/2}\,Z_2$$

- Combinations of Z-factors fixed by $SU(N)$ gauge invariance of QCD
  - Slavnov-Taylor (or Ward) identites

$$Z_g\,(Z_3)^{1/2} \;=\; \frac{Z_1}{Z_3} \;=\; \frac{Z_{1F}}{Z_2}$$

Renormalized Lagrangian

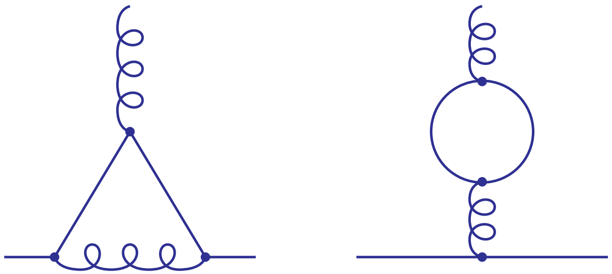
- Construction of renormalized QCD Lagrangian $\mathcal{L}^{\rm ren}$ with rescaled fields, parameters. etc.
- $\mathcal{L}^{\rm bare}$ decomposed into $\mathcal{L}^{\rm ren}$ and counter term $\mathcal{L}^{\rm ct}$

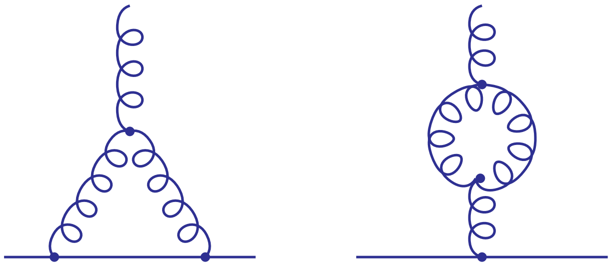
$$\mathcal{L}^{\rm bare}\left(\psi^{\rm b}, \bar{\psi}^{\rm b}, A_\mu^{\rm b}\right) \;=\; \mathcal{L}^{\rm ren}\left(\psi^{\rm r}, \bar{\psi}^{\rm r}, A_\mu^{\rm r}\right) + \mathcal{L}^{\rm ct}$$

  - $\mathcal{L}^{\rm ct}$ contains all parameters with factors $(Z_i - 1)$
  - ultraviolet divergences absorbed by $\mathcal{L}^{\rm ct}$

Running coupling

- Effective coupling constant α_s depends on resolution
- QCD distinguished by self-interaction of gluons; e.g. vertex corrections





– screening (like in QED)

– anti-screening (color charge of g)

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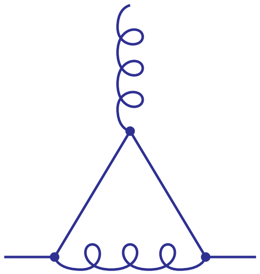

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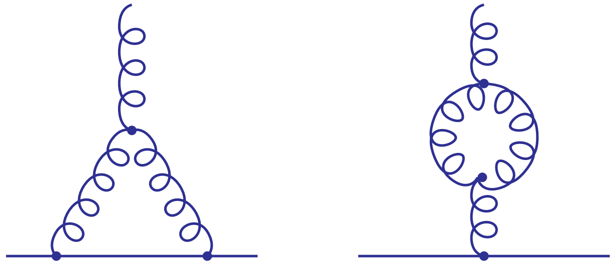


– anti-screening (color charge of g)

- Scale dependence governed by β-function of QCD

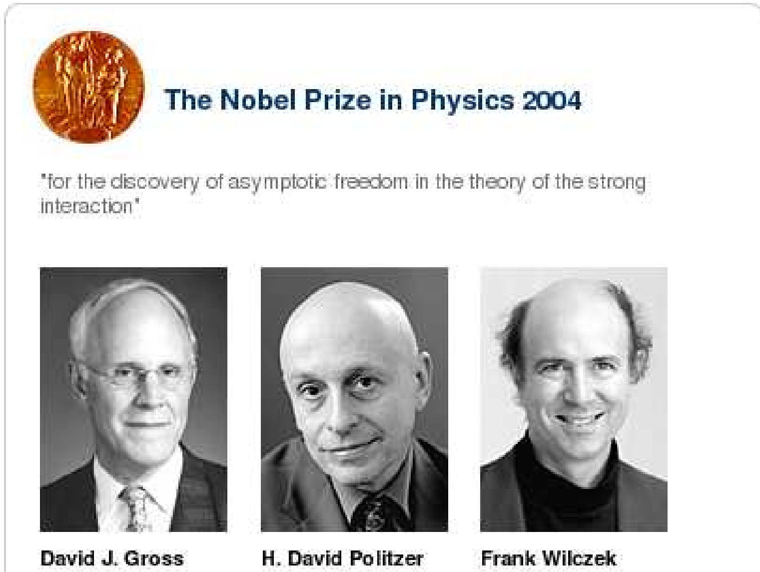
$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu) \;=\; \beta\left(\alpha_s\right) \;=\; -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 - \beta_3 \alpha_s^5 - \ldots$$

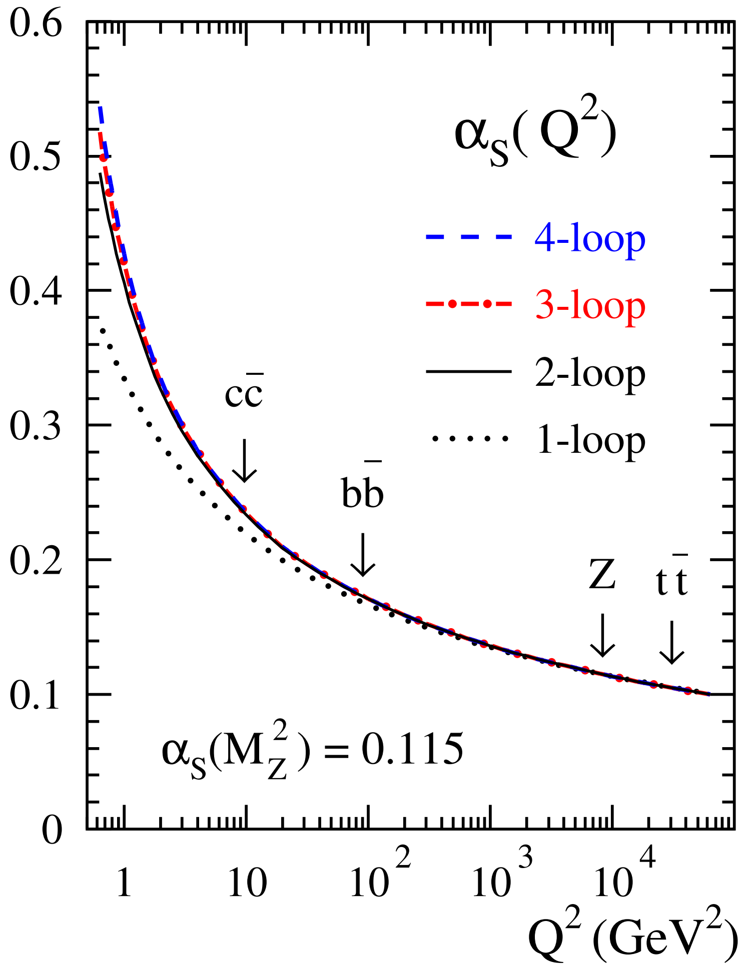
  - QCD β-function has negative sign
  - perturbative expansion with coefficients $\beta_0, \beta_1, \beta_2, \ldots$

$$\beta_0 \;=\; \frac{1}{4\pi}\left(\frac{11}{3} C_A - \frac{2}{3} n_f\right) \;=\; \frac{1}{4\pi}\left(7\right) \qquad \text{(for } n_f = 6\text{)}$$

Asymptotic freedom

- Perturbative solution of QCD β-function to five loops
  Baikov, Chetyrkin, Kühn '16; Herzog, Ruijl, Ueda, Vermaseren, Vogt '17; Luthe, Maier, Marquard, Schröder '17; Chetyrkin, Falcioni, Herzog, Vermaseren '17
  - very good convergence of perturbative series even at low scales (but $\alpha_s \gg \alpha_{\text{QED}}$)





Strong coupling constant (I)

Measurements of strong coupling

- Couplings and masses are formal parameters of the theory
  - $\alpha_s = g_s^2/(4\pi)$ and m_q are no observables
- Recall classical part of QCD Lagrangian

$$\mathcal{L} \quad = \quad -\frac{1}{4} F_{\mu\nu}^a F_b^{\mu\nu} + \sum_{\text{flavors}} \bar{q}_i \left(\mathrm{i} \not{D} - m_q\right)_{ij} q_j$$

  - field strength tensor $F_{\mu\nu}^a$ and matter fields $q_i, \bar{q}_j$
  - covariant derivative $D_{\mu,ij} = \partial_\mu \delta_{ij} + \mathrm{i} g_s \left(t_a\right)_{ij} A_\mu^a$
- Parameters of Lagrangian have no unique physical interpretation
  - radiative corrections require definition of renormalization scheme

Challenge

- Suitable observables for measurements of α_s, m_q, ...
  - comparison of theory predictions and experimental data

Strong coupling constant (II)

Essential facts

- $\alpha_s(M_Z)$ from e^+e^- data high
- $\alpha_s(M_Z)$ from DIS data low
- NLO QCD theory
- World average 1992
  $\alpha_s(M_Z) = 0.117 \pm 0.004$

Bethke, Catani CERN TH-6484/92

	Process	Ref.	Q [GeV]	$\alpha_s(Q)$	$\alpha_s(M_{Z^0})$	$\Delta\alpha_s(M_{Z^0})$ exp.	$\Delta\alpha_s(M_{Z^0})$ theor.	order of perturb.
1	R_τ [LEP]	[7-10]	1.78	$0.318 ^{+0.048}_{-0.039}$	$0.117 ^{+0.006}_{-0.005}$	$^{+0.003}_{-0.004}$	$^{+0.005}_{-0.004}$	NNLO
2	R_τ [world]	[2]	1.78	0.32 ± 0.04	$0.118 ^{+0.004}_{-0.006}$	–	–	NNLO
3	DIS [ν]	[3]	5.0	$0.193 ^{+0.019}_{-0.018}$	$0.111 ^{+0.006}_{-0.007}$	$^{+0.004}_{-0.006}$	0.004	NLO
4	DIS [μ]	[12]	7.1	0.180 ± 0.014	0.113 ± 0.005	0.003	0.004	NLO
5	$J/\Psi, \Upsilon$ decay	[4]	10.0	$0.167 ^{+0.015}_{-0.011}$	$0.113 ^{+0.007}_{-0.005}$	–	–	NLO
6	e^+e^- [σ_{had}]	[14]	34.0	0.163 ± 0.022	0.135 ± 0.015	–	–	NNLO
7	e^+e^- [shapes]	[15]	35.0	0.14 ± 0.02	0.119 ± 0.014	–	–	NLO
8	$p\bar{p} \to b\bar{b}X$	[11]	20.0	$0.136 ^{+0.025}_{-0.024}$	$0.108 ^{+0.015}_{-0.014}$	0.006	$^{+0.014}_{-0.013}$	NLO
9	$p\bar{p} \to W$ *jets*	[13]	80.6	0.123 ± 0.027	0.121 ± 0.026	0.018	0.020	NLO
10	$\Gamma(Z^0 \to$ had.)	[5]	91.2	0.133 ± 0.012	0.133 ± 0.012	0.012	$^{+0.003}_{-0.001}$	NNLO
11	Z^0 ev. shapes							
	ALEPH	[7]	91.2	$0.119 ^{+0.008}_{-0.010}$		–	–	NLO
	DELPHI	[8]	91.2	0.113 ± 0.007		0.002	0.007	NLO
	L3	[9]	91.2	0.118 ± 0.010		–	–	NLO
	OPAL	[10]	91.2	$0.122 ^{+0.006}_{-0.005}$		0.001	$^{+0.006}_{-0.005}$	NLO
	SLD	[6]	91.2	$0.120 ^{+0.015}_{-0.013}$		0.009	$^{+0.012}_{-0.009}$	NLO
	Average	[6-10]	91.2		0.119 ± 0.006	0.001	0.006	NLO
12	Z^0 ev. shapes							
	ALEPH	[7]	91.2	0.125 ± 0.005		0.002	0.004	resum.
	DELPHI	[8]	91.2	0.122 ± 0.006		0.002	0.006	resum.
	L3	[9]	91.2	0.126 ± 0.009		0.003	0.008	resum.
	OPAL	[10]	91.2	$0.122 ^{+0.003}_{-0.006}$		0.001	$^{+0.003}_{-0.006}$	resum.
	Average	[7-10]	91.2		0.123 ± 0.005	0.001	0.005	resum.

Table 1: Summary of measurements of α_s. For details see text.

2

Strong coupling constant (III)

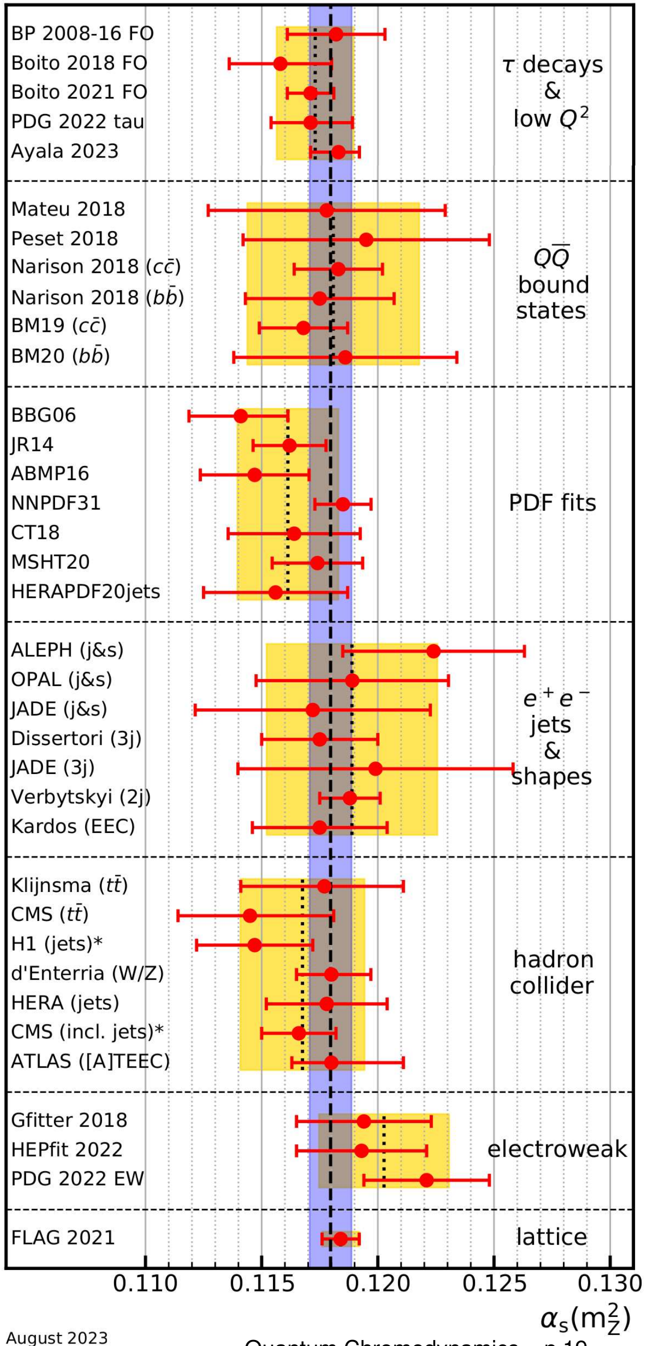
α_s 2024

- $\alpha_s(M_Z)$ from e^+e^- data high
- $\alpha_s(M_Z)$ from DIS data low
- NNLO QCD theory + lattice

FCC-ee

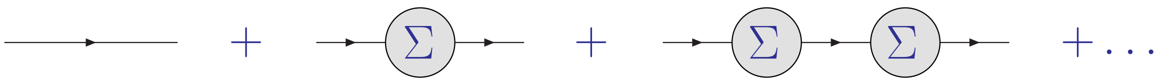
- Potential of FCC-ee:
  $\alpha_s(M_Z)$ at 1% precision

PDG 2024



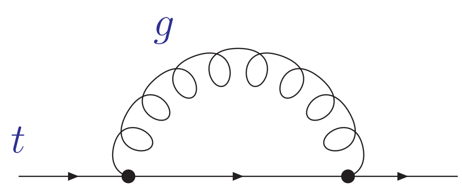
Quark mass renormalization

- Heavy-quark self-energy $\Sigma(p, m_q)$



$$= \frac{i}{\not{p} - m_q - \Sigma(p, m_q)}$$

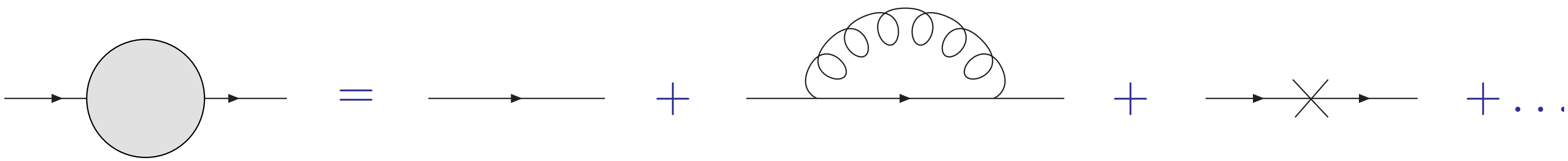
QCD



- QCD corrections to self-energy $\Sigma(p, m_q)$
  - dimensional regularization $D = 4 - 2\epsilon$
  - one-loop: UV divergence $1/\epsilon$ (Laurent expansion)

$$\Sigma^{(1),\text{bare}}(p, m_q) = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{m_q^2}\right)^{\epsilon} \left\{ (\not{p} - m_q) \left(-C_F \frac{1}{\epsilon} + \text{fin.} \right) + m_q \left(3C_F \frac{1}{\epsilon} + \text{fin.} \right) \right\}$$

- Relate bare and renormalized mass parameter $m_q^{\text{bare}} = m_q^{\text{ren}} + \delta m_q$



$\Sigma^{\text{ren}}(p, m_q)$ $\qquad$ $(Z_\psi - 1)\not{p} - (Z_m - 1)m_q$

Mass renormalization scheme

Pole mass

- Based on (unphysical) concept of top-quark being a free parton
  - m_q^{ren} coincides with pole of propagator at each order

$$\not{p} - m_q - \Sigma(p, m_q)\Big|_{\not{p}=m_q} \to \not{p} - m_q^{\text{pole}}$$

- Definition of pole mass ambiguous up to corrections $\mathcal{O}(\Lambda_{QCD})$
  - heavy-quark self-energy $\Sigma(p, m_q)$ receives contributions from regions of all loop momenta – also from momenta of $\mathcal{O}(\Lambda_{QCD})$

$\overline{MS}$ *scheme*

- $\overline{MS}$ mass definition
  - one-loop minimal subtraction

$$\delta m_q^{(1)} = m_q \frac{\alpha_s}{4\pi} 3C_F \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

- $\overline{MS}$ scheme induces scale dependence: $m(\mu)$

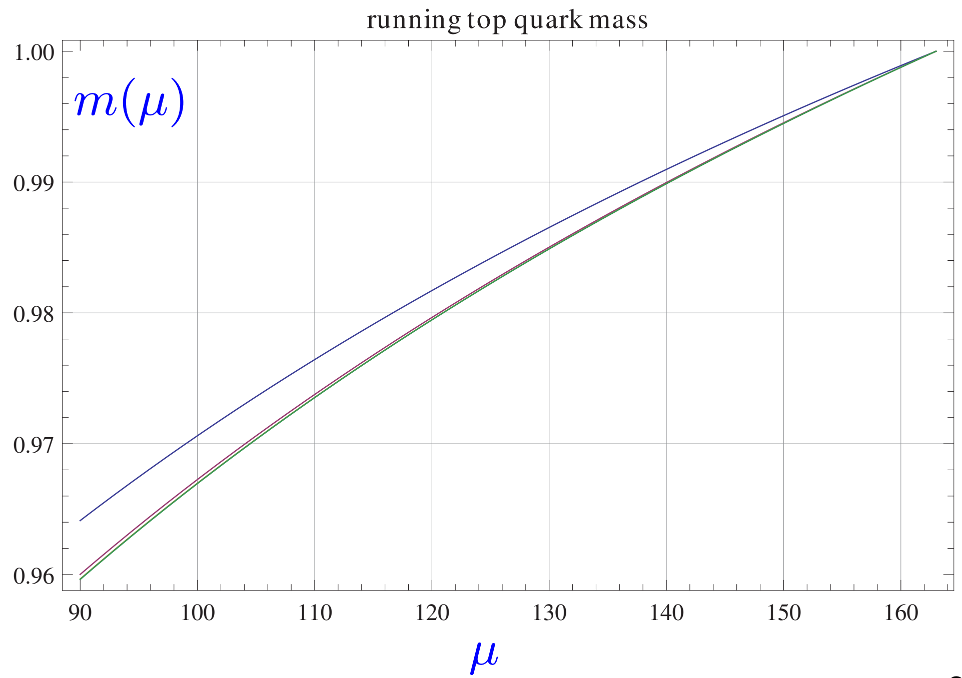
Running quark mass

Scale dependence

- Renormalization group equation for scale dependence
  - mass anomalous dimension γ known to five loops
    Baikov, Chetyrkin, Kühn ‘14, Luthe, Maier, Marquard, Schröder ‘17

$$\left(\mu^2 \frac{\partial}{\partial\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right) m(\mu) \;=\; \gamma(\alpha_s) m(\mu)$$

- Plot mass ratio $m_t(163\text{GeV})/m_t(\mu)$



Scheme transformations

- Conversion between different renormalization schemes possible in perturbation theory
- Relation for pole mass and $\overline{MS}$ mass
  - known to four loops in QCD Gray, Broadhurst, Gräfe, Schilcher ‘90; Chetyrkin, Steinhauser ‘99; Melnikov, v. Ritbergen ‘99; Marquard, Smirnov, Smirnov, Steinhauser ‘15
  - example: one-loop QCD

$$m^{\text{pole}} = m(\mu) \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} \left(\frac{4}{3} + \ln \left(\frac{\mu^2}{m(\mu)^2} \right) \right) + \ldots \right\}$$

Decoupling

In a nut-shell

- QCD with different number of quarks can be related $\longrightarrow$ matching of two distinct theories
- Heavy quarks can be decoupled in limit $m_q \to \infty$ Appelquist, Carrazone '74
- Consider QCD parameters in both theories and match at scale μ
  - n_l light flavors + n_h heavy quarks of masses m_q at low scales
  - $n_l + n_h$ light flavors at high scales
- Example: running coupling constant

$$\alpha_s^{n_l} \longrightarrow \alpha_s^{(n_l+n_h)}$$

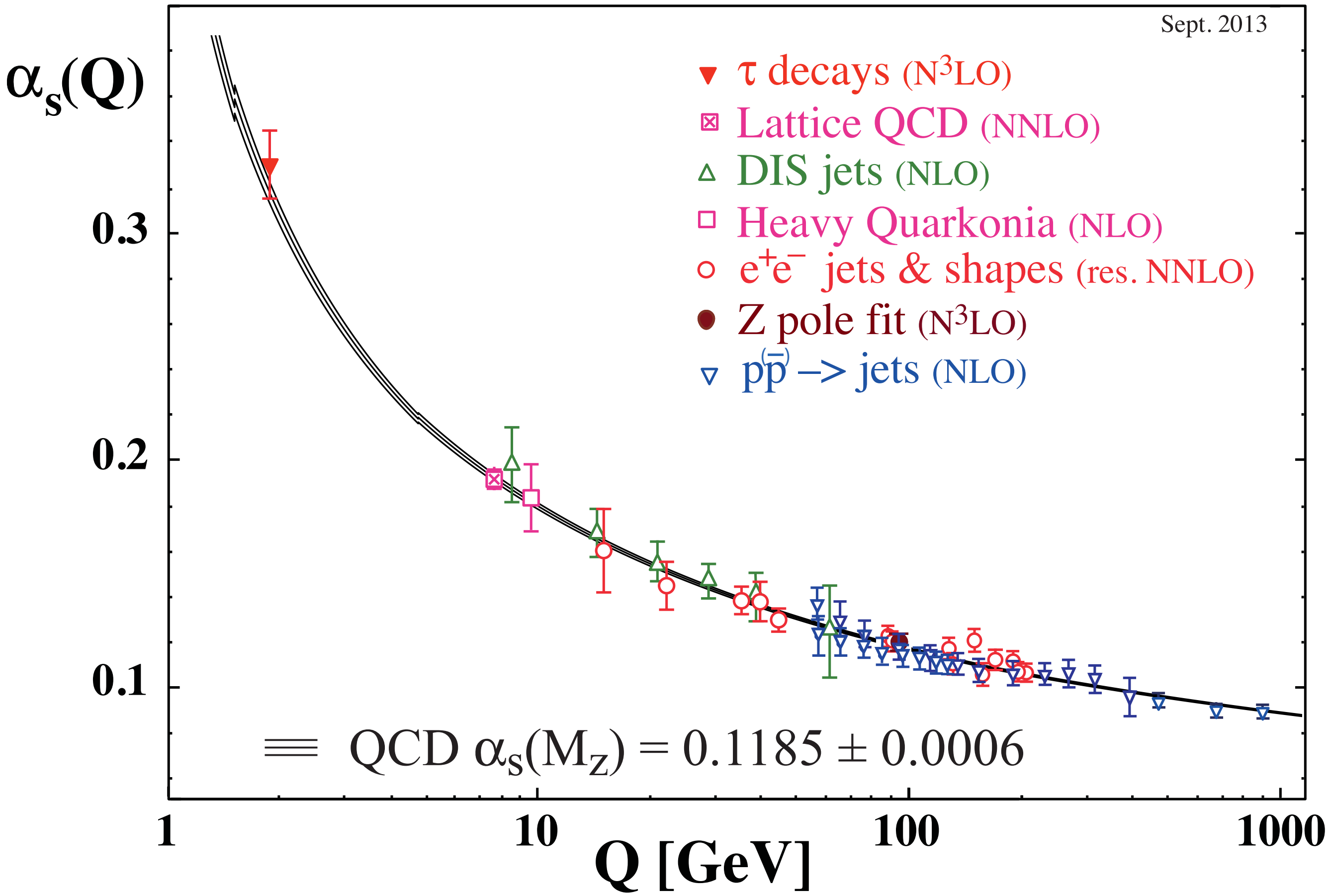
$\overline{\mathrm{MS}}$ scheme

- Decoupling theorem in the $\overline{\mathrm{MS}}$-scheme not true in naive sense
  - mass effects not $1/m_q$ suppressed in theory with n_l light and n_h heavy flavors
  - anomalous dimensions exhibit discontinuities at flavor thresholds
- Decoupling constants in the $\overline{\mathrm{MS}}$ scheme
  Larin, van Ritbergen, Vermaseren '94; Chetyrkin, Kniehl, Steinhauser '97

Strong coupling with flavor thresholds

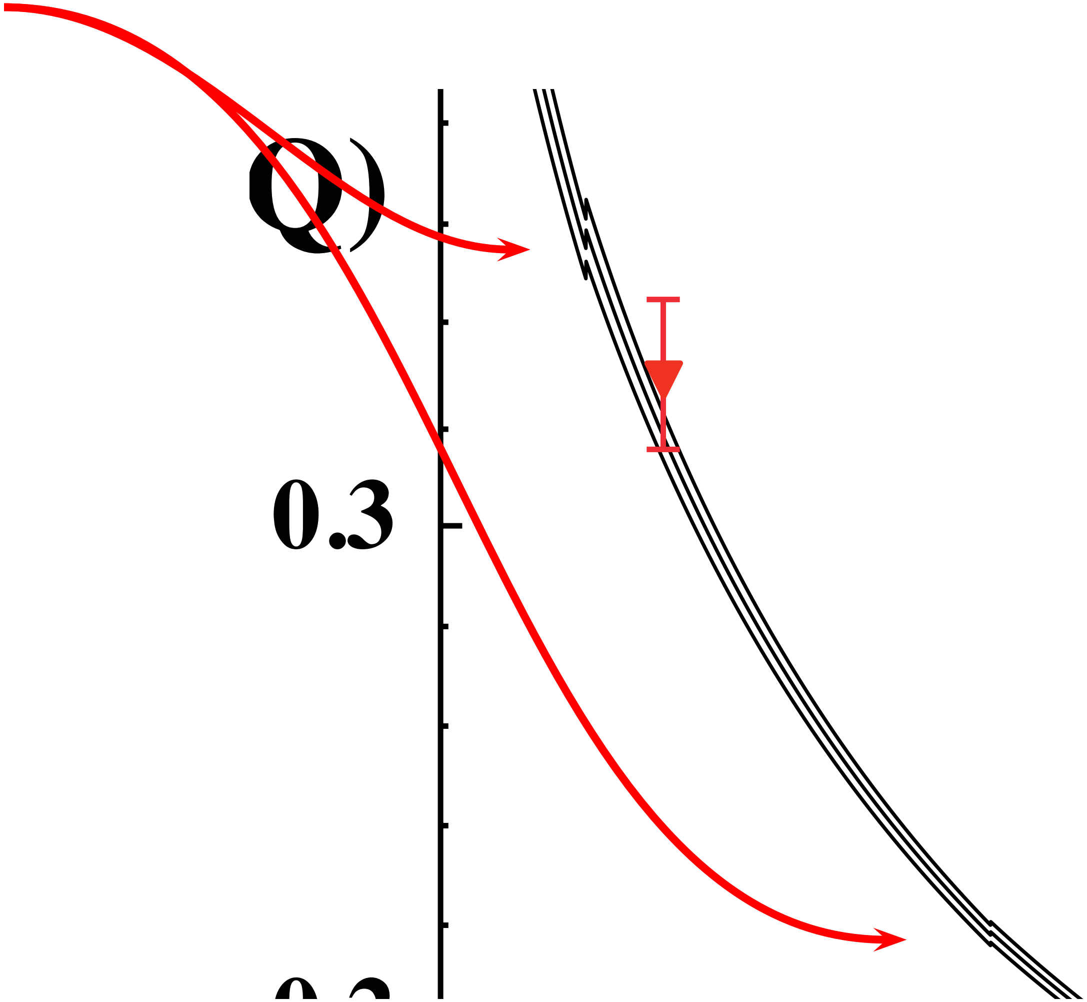
- Solution of QCD β-function for $\alpha_s^{n_l} \longrightarrow \alpha_s^{(n_l+n_h)}$
  - discontinuities for $n_f = 3 \rightarrow n_f = 4 \rightarrow n_f = 5$
- Big picture

Bethke for PDG 2014



Strong coupling with flavor thresholds

- Solution of QCD β-function for $\alpha_s^{n_l} \longrightarrow \alpha_s^{(n_l+n_h)}$
  - discontinuities for $n_f = 3 \rightarrow n_f = 4 \rightarrow n_f = 5$
- Zoom



Summary (part I)

QCD: the gauge theory of the strong interaction

- Quarks and gluons as classical degrees of freedom
- Quantum corrections determine dynamical properties
  - scale dependence of observables
  - running coupling constant and asymptotic freedom
- Renormalization required by quantum corrections
  - subtraction of ultraviolet singularities
  - definition of renormalization scheme
  - parameters of Lagrangian are not observables $\alpha_s, m_q, \dots$