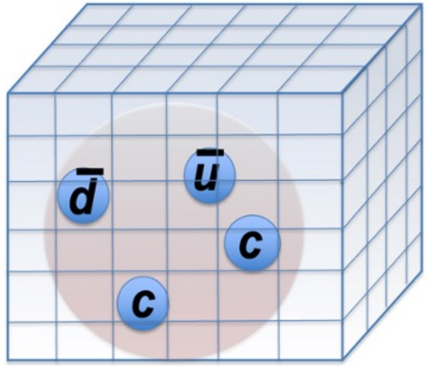
Exotic hadrons: A Lattice QCD perspective



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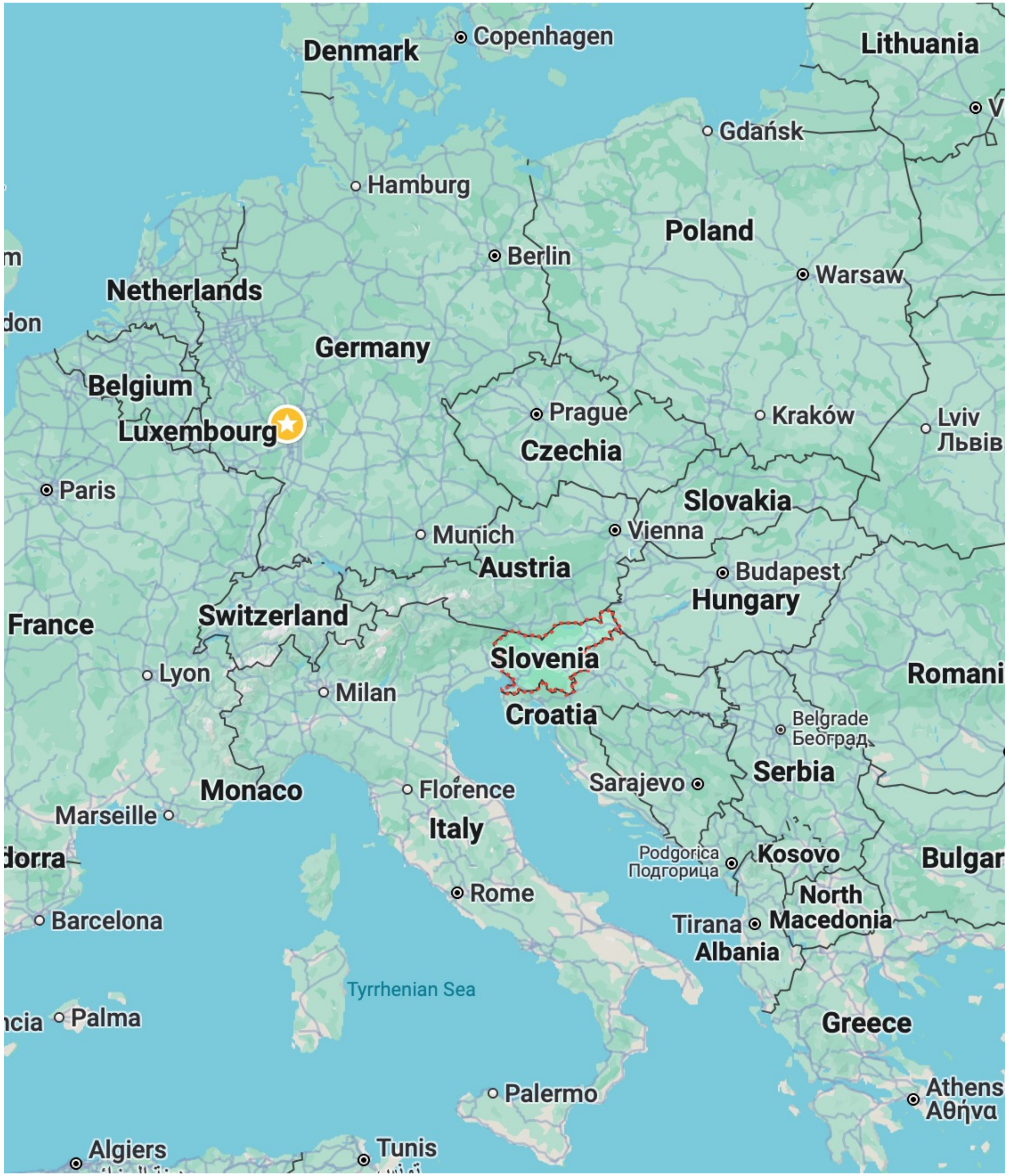
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Tirana
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Palermo
Athens
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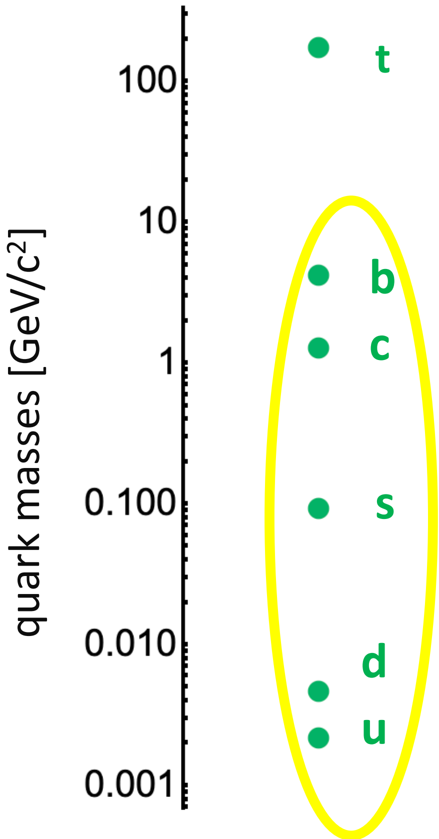
Tunis

1996

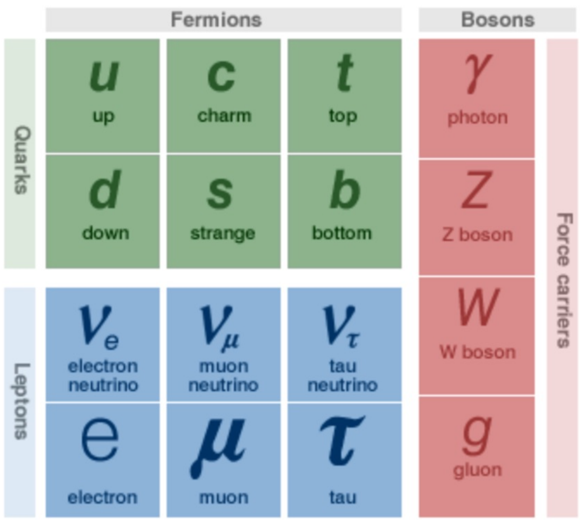
1996

Main players : Quarks and strong interaction



$1\ eV \simeq 1.6 \cdot 10^{-19}\ J$

Fundamental particles:

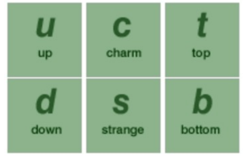


H
Higgs b.

Fundamental forces:

- electromagnetic
- weak
- strong
- gravitational

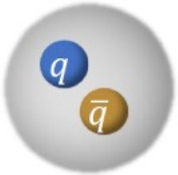
Conventional and exotic hadrons

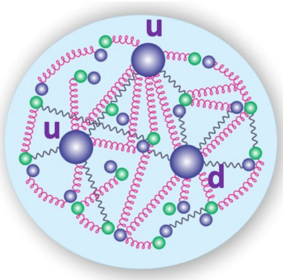


Minimal valence content

baryon

meson

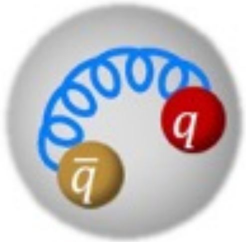




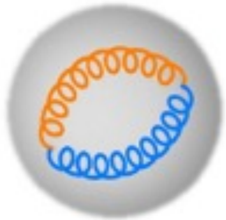
tetraquark

pentaquark

hybrid meson



glueball



+

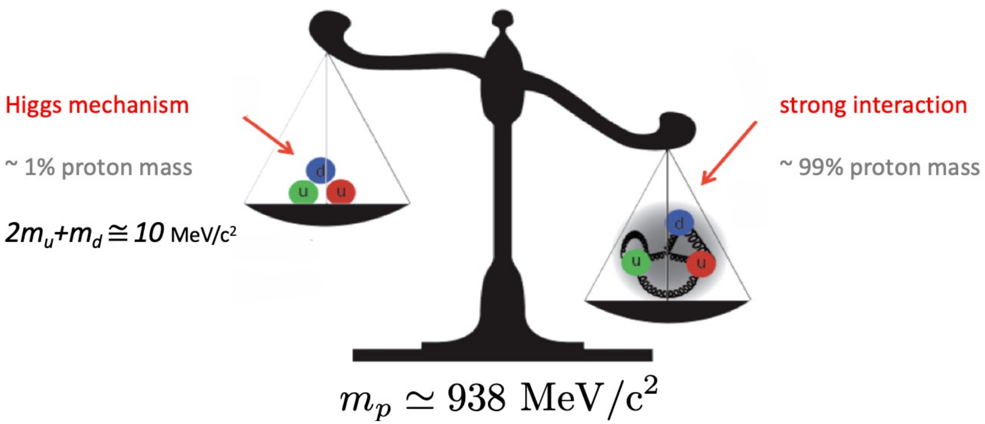
cypto exotic

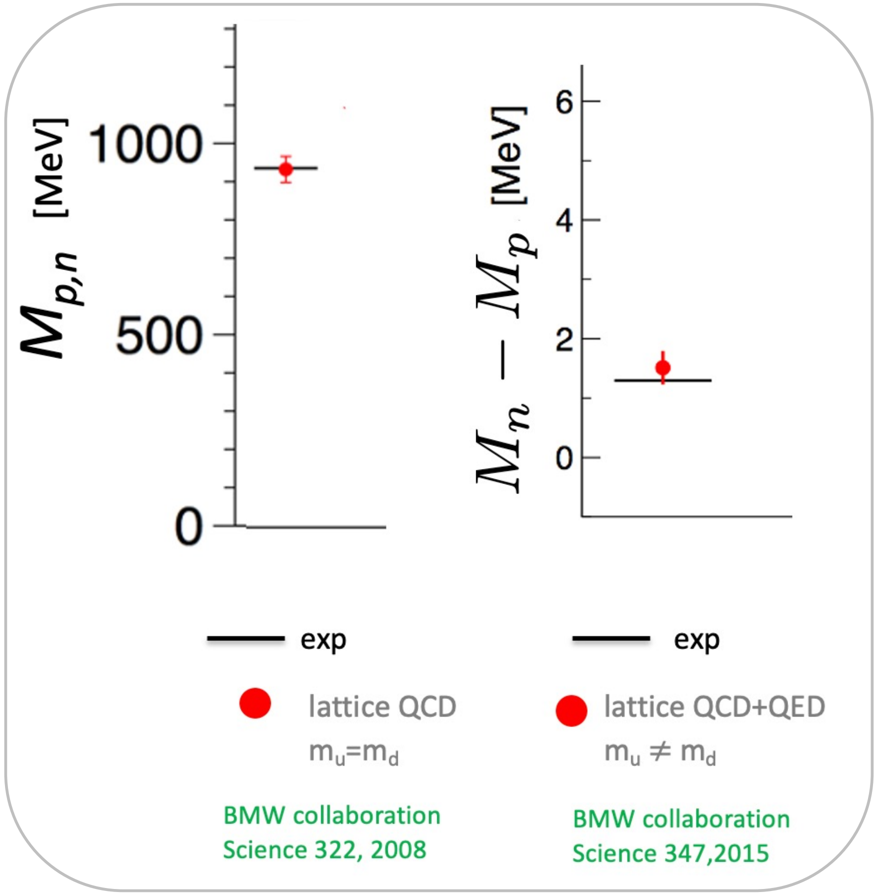
(not within quark model of qqq, $\bar{q}q$)

Terminology in this talk: tetra(penta) quarks indicate just the number of valence quarks in the state; it is not meant to say anything on how quarks are clustered in them

(Mostly) conventional hadrons

Mass of proton and neutron





(Mostly) conventional baryons qqq

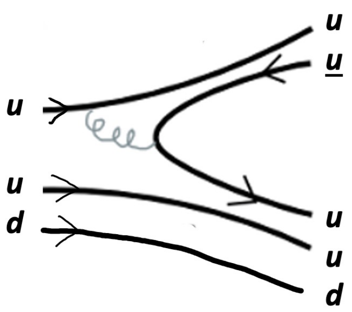
p	$1/2^+$	****
n	$1/2^+$	****
$N(1440)$	$1/2^+$	****
$N(1520)$	$3/2^-$	****
$N(1535)$	$1/2^-$	****
$N(1650)$	$1/2^-$	****
$N(1675)$	$5/2^-$	****
$N(1680)$	$5/2^+$	****
$N(1700)$	$3/2^-$	***
$N(1710)$	$1/2^+$	****
$N(1720)$	$3/2^+$	****
$N(1860)$	$5/2^+$	**
$N(1875)$	$3/2^-$	***
was $N(2080)$		
$N(1880)$	$1/2^+$	***
$N(1895)$	$1/2^-$	****
was $N(2090)$		
$N(1900)$	$3/2^+$	****
$N(1990)$	$7/2^+$	**
$N(2000)$	$5/2^+$	**
was $N(1900)$		
$N(2040)$	$3/2^+$	*
$N(2060)$	$5/2^-$	***
was $N(2200)$		
$N(2100)$	$1/2^+$	***
$N(2120)$	$3/2^-$	***
$N(2190)$	$7/2^-$	****
$N(2220)$	$9/2^+$	****
$N(2250)$	$9/2^-$	****
$N(2300)$	$1/2^+$	**
$N(2570)$	$5/2^-$	**
$N(2600)$	$11/2^-$	***
$N(2700)$	$13/2^+$	**

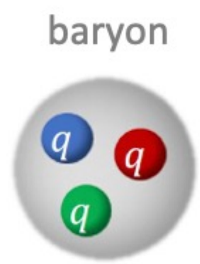
$\Delta(1232)$	$3/2^+$	****
$\Delta(1600)$	$3/2^+$	****
$\Delta(1620)$	$1/2^-$	****
$\Delta(1700)$	$3/2^-$	****
$\Delta(1750)$	$1/2^+$	*
$\Delta(1900)$	$1/2^-$	***
$\Delta(1905)$	$5/2^+$	****
$\Delta(1910)$	$1/2^+$	****
$\Delta(1920)$	$3/2^+$	***
$\Delta(1930)$	$5/2^-$	***
$\Delta(1940)$	$3/2^-$	**
$\Delta(1950)$	$7/2^+$	****
$\Delta(2000)$	$5/2^+$	**
$\Delta(2150)$	$1/2^-$	*
$\Delta(2200)$	$7/2^-$	***
$\Delta(2300)$	$9/2^+$	**
$\Delta(2350)$	$5/2^-$	*
$\Delta(2390)$	$7/2^+$	*
$\Delta(2400)$	$9/2^-$	**
$\Delta(2420)$	$11/2^+$	****
$\Delta(2750)$	$13/2^-$	**
$\Delta(2950)$	$15/2^+$	**
Λ	$1/2^+$	****
$\Lambda(1380)$	$1/2^-$	**
$\Lambda(1405)$	$1/2^-$	****
$\Lambda(1520)$	$3/2^-$	****
$\Lambda(1600)$	$1/2^+$	****
$\Lambda(1670)$	$1/2^-$	****
$\Lambda(1690)$	$3/2^-$	****
$\Lambda(1710)$	$1/2^+$	*
$\Lambda(1800)$	$1/2^-$	***
$\Lambda(1810)$	$1/2^+$	***
$\Lambda(1820)$	$5/2^+$	****
$\Lambda(1830)$	$5/2^-$	****
$\Lambda(1890)$	$3/2^+$	****
$\Lambda(2000)$	$1/2^-$	*

Σ^+	$1/2^+$	****
Σ^0	$1/2^+$	****
Σ^-	$1/2^+$	****
$\Sigma(1385)$	$3/2^+$	****
$\Sigma(1580)$	$3/2^-$	*
$\Sigma(1620)$	$1/2^-$	*
$\Sigma(1660)$	$1/2^+$	***
$\Sigma(1670)$	$3/2^-$	****
$\Sigma(1750)$	$1/2^-$	***
$\Sigma(1775)$	$5/2^-$	****
$\Sigma(1780)$	$3/2^+$	*
was $\Sigma(1730)$		
$\Sigma(1880)$	$1/2^+$	**
$\Sigma(1900)$	$1/2^-$	**
$\Sigma(1910)$	$3/2^-$	***
was $\Sigma(1940)$		
$\Sigma(1915)$	$5/2^+$	****
$\Sigma(1940)$	$3/2^+$	*
$\Sigma(2010)$	$3/2^-$	*
was $\Sigma(2000)$		
$\Sigma(2030)$	$7/2^+$	****
$\Sigma(2070)$	$5/2^+$	*
$\Sigma(2080)$	$3/2^+$	*
$\Sigma(2100)$	$7/2^-$	*
$\Sigma(2110)$	$1/2^-$	*
was $\Sigma(2160)$		
$\Sigma(2230)$	$3/2^+$	*
$\Sigma(2250)$		**
$\Sigma(2455)$		*
$\Sigma(2620)$		*
$\Sigma(3000)$		*
$\Sigma(3170)$		*

Ξ^0	$1/2^+$	****
Ξ^-	$1/2^+$	****
$\Xi(1530)$	$3/2^+$	****
$\Xi(1620)$		**
$\Xi(1690)$		***
$\Xi(1820)$	$3/2^-$	***
$\Xi(1950)$		***
$\Xi(2030)$	$\frac{5}{2}^?$	***
$\Xi(2120)$		*
$\Xi(2250)$		**
$\Xi(2370)$		**
$\Xi(2500)$		*
Ω^-	$3/2^+$	****
$\Omega(2012)^-$	$?^-$	***
$\Omega(2250)^-$		***
$\Omega(2380)^-$		**
$\Omega(2470)^-$		**

Λ_c^+	$1/2^+$	****
$\Lambda_c(2595)^+$	$1/2^-$	***
$\Lambda_c(2625)^+$	$3/2^-$	***
$\Lambda_c(2765)^+$ or $\Sigma_c(2765)$		*
$\Lambda_c(2860)^+$	$3/2^+$	***
$\Lambda_c(2880)^+$	$5/2^+$	***
$\Lambda_c(2910)^+$		*
$\Lambda_c(2940)^+$	$3/2^-$	***
$\Sigma_c(2455)$	$1/2^+$	****
$\Sigma_c(2520)$	$3/2^+$	***
$\Sigma_c(2800)$		***
Ξ_c^+	$1/2^+$	***
Ξ_c^0	$1/2^+$	****
$\Xi_c'^+$	$1/2^+$	***
$\Xi_c'^0$	$1/2^+$	***
$\Xi_c(2645)$	$3/2^+$	***
$\Xi_c(2790)$	$1/2^-$	***
$\Xi_c(2815)$	$3/2^-$	***
$\Xi_c(2882)$		*
$\Xi_c(2923)$		**
$\Xi_c(2930)$		**
$\Xi_c(2970)$	$1/2^+$	***
was $\Xi_c(2980)$		
$\Xi_c(3055)$		***
$\Xi_c(3080)$		***
$\Xi_c(3123)$		*
Ω_c^0	$1/2^+$	***
$\Omega_c(2770)^0$	$3/2^+$	***
$\Omega_c(3000)^0$		***
$\Omega_c(3050)^0$		***
$\Omega_c(3065)^0$		***
$\Omega_c(3090)^0$		***
$\Omega_c(3120)^0$		***
$\Omega_c(3185)^0$		***
$\Omega_c(3327)^0$		***





(Mostly) conventional mesons $\bar{q}q$ (including some non-qq states)

LIGHT UNFLAVORED ($S = C = B = 0$)

	$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$
• π^0	$1^-(0^{-+})$
• η	$0^+(0^{-+})$
• $f_0(500)$	$0^+(0^{++})$
aka σ; was	
$f_0(600)$,	
$f_0(400-1200)$	
• $\rho(770)$	$1^+(1^{--})$
• $\omega(782)$	$0^-(1^{--})$
• $\eta'(958)$	$0^+(0^{-+})$
• $f_0(980)$	$0^+(0^{++})$
• $a_0(980)$	$1^-(0^{++})$
• $\phi(1020)$	$0^-(1^{--})$
• $h_1(1170)$	$0^-(1^{+-})$
• $b_1(1235)$	$1^+(1^{+-})$
• $a_1(1260)$	$1^-(1^{++})$
• $f_2(1270)$	$0^+(2^{++})$
• $f_1(1285)$	$0^+(1^{++})$
• $\eta(1295)$	$0^+(0^{-+})$
• $\pi(1300)$	$1^-(0^{-+})$
• $a_2(1320)$	$1^-(2^{++})$
• $f_0(1370)$	$0^+(0^{++})$
$\pi_1(1400)$	$1^-(1^{-+})$
• $\eta(1405)$	$0^+(0^{-+})$
• $h_1(1415)$	$0^-(1^{+-})$
• $f_1(1420)$	$0^+(1^{++})$
• $\omega(1420)$	$0^-(1^{--})$
$f_2(1430)$	$0^+(2^{++})$
• $a_0(1450)$	$1^-(0^{++})$
• $\rho(1450)$	$1^+(1^{--})$
• $\eta(1475)$	$0^+(0^{-+})$
• $f_0(1500)$	$0^+(0^{++})$
$f_1(1510)$	$0^+(1^{++})$
• $f_2'(1525)$	$0^+(2^{++})$
• $f_2(1565)$	$0^+(2^{++})$
$\rho(1570)$	$1^+(1^{--})$
$h_1(1595)$	$0^-(1^{+-})$
• $\pi_1(1600)$	$1^-(1^{-+})$
• $a_1(1640)$	$1^-(1^{++})$
$f_2(1640)$	$0^+(2^{++})$
• $\eta_2(1645)$	$0^+(2^{-+})$
• $\omega(1650)$	$0^-(1^{--})$
• $\omega_3(1670)$	$0^-(3^{--})$
• $\pi_2(1670)$	$1^-(2^{-+})$
• $\phi(1680)$	$0^-(1^{--})$
• $\rho_3(1690)$	$1^+(3^{--})$

	$I^G(J^{PC})$
• $\rho(1700)$	$1^+(1^{--})$
• $a_2(1700)$	$1^-(2^{++})$
$a_0(1710)$	$1^-(0^{++})$
• $f_0(1710)$	$0^+(0^{++})$
$X(1750)$	$?^-(1^{--})$
$\eta(1760)$	$0^+(0^{-+})$
$f_0(1770)$	$0^+(0^{++})$
• $\pi(1800)$	$1^-(0^{-+})$
$f_2(1810)$	$0^+(2^{++})$
$X(1835)$	$0^+(0^{-+})$
• $\phi_3(1850)$	$0^-(3^{--})$
$\eta_1(1855)$	$0^+(1^{-+})$
• $\eta_2(1870)$	$0^+(2^{-+})$
• $\pi_2(1880)$	$1^-(2^{-+})$
$\rho(1900)$	$1^+(1^{--})$
$f_2(1910)$	$0^+(2^{++})$
$a_0(1950)$	$1^-(0^{++})$
• $f_2(1950)$	$0^+(2^{++})$
• $a_4(1970)$	$1^-(4^{++})$
$\rho_3(1990)$	$1^+(3^{--})$
$\pi_2(2005)$	$1^-(2^{-+})$
• $f_2(2010)$	$0^+(2^{++})$
• $f_0(2020)$	$0^+(0^{++})$
• $f_4(2050)$	$0^+(4^{++})$
$\pi_2(2100)$	$1^-(2^{-+})$
$f_0(2100)$	$0^+(0^{++})$
$f_2(2150)$	$0^+(2^{++})$
$\rho(2150)$	$1^+(1^{--})$
• $\phi(2170)$	$0^-(1^{--})$
$f_0(2200)$	$0^+(0^{++})$
$f_J(2220)$	$0^+(2^{++}$ or $4^{++})$
$\omega(2220)$	$0^-(1^{--})$
$\eta(2225)$	$0^+(0^{-+})$
$\rho_3(2250)$	$1^+(3^{--})$
• $f_2(2300)$	$0^+(2^{++})$
$f_4(2300)$	$0^+(4^{++})$
$f_0(2330)$	$0^+(0^{++})$
• $f_2(2340)$	$0^+(2^{++})$
$\rho_5(2350)$	$1^+(5^{--})$
$X(2370)$	$?^?(?^{??})$
$f_0(2470)$	$0^+(0^{++})$
$f_6(2510)$	$0^+(6^{++})$

STRANGE ($S = \pm1$, $C = B = 0$)

	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
• $K_0^*(700)$	$1/2(0^+)$
aka κ; was	
$K_0^*(800)$	
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K_0^*(1430)$	$1/2(0^+)$
• $K_2^*(1430)$	$1/2(2^+)$
• $K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
• $K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K_3^*(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
• $K_0^*(1950)$	$1/2(0^+)$
• $K_2^*(1980)$	$1/2(2^+)$
• $K_4^*(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$
$K_5^*(2380)$	$1/2(5^-)$
$K_4(2500)$	$1/2(4^-)$
$K(3100)$	$?^?(?^{??})$
aka $K_J^?(3100)$	

CHARMED ($C = \pm1$)

	$I(J^P)$
• $D^\pm$	$1/2(0^-)$
• D^0	$1/2(0^-)$
• $D^*(2007)^0$	$1/2(1^-)$
• $D^*(2010)^\pm$	$1/2(1^-)$
• $D_0^*(2300)$	$1/2(0^+)$
was $D_0^*(2400)$	
• $D_1(2420)$	$1/2(1^+)$
• $D_1(2430)^0$	$1/2(1^+)$
• $D_2^*(2460)$	$1/2(2^+)$
$D_0(2550)^0$	$1/2(0^-)$
$D_1^*(2600)^0$	$1/2(1^-)$
was $D_J^*(2600)$	
$D^*(2640)^\pm$	$1/2(?^?)$
$D_2(2740)^0$	$1/2(2^-)$
was $D(2740)^0$	
• $D_3^*(2750)$	$1/2(3^-)$

CHARMED, STRANGE ($C = \pm1$, $S = \pm1$) (including possibly non-$q\bar{q}$ states)

	$I(J^P)$
• $D_s^\pm$	$0(0^-)$
• $D_s^{*\pm}$	$0(1^-)$
• $D_{s0}^*(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}^*(2573)$	$0(2^+)$
$D_{s0}(2590)^+$	$0(0^-)$
• $D_{s1}^*(2700)^\pm$	$0(1^-)$
$D_{s1}^*(2860)^\pm$	$0(1^-)$
was $D_{sJ}^*(2860)$	
• $D_{s3}^*(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

BOTTOM ($B = \pm1$)

• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B^0_s/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)$	$1/2(1^+)$
$B_J^*(5732)$	$?(?^?)$
• $B_2^*(5747)$	$1/2(2^+)$
$B_J(5840)$	$1/2(?^?)$
• $B_J(5970)$	$1/2(?^?)$

BOTTOM, STRANGE ($B = \pm1$, $S = \mp1$)

• B_s^0	$0(0^-)$
• B_s^*	$0(1^-)$
• $B_{s1}(5830)^0$	$0(1^+)$
• $B_{s2}^*(5840)^0$	$0(2^+)$
$B_{sJ}^*(5850)$	$?(?^?)$
$B_{sJ}(6063)^0$	$0(?^?)$
$B_{sJ}(6114)^0$	$0(?^?)$

BOTTOM, CHARMED ($B = C = \pm1$)

• B_c^+	$0(0^-)$
• $B_c(2S)^\pm$	$0(0^-)$

$c\bar{c}$ (including possibly non-$q\bar{q}$ states)

• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$0^-(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$

$b\bar{b}$ (including possibly non-$q\bar{q}$ states)

	$I^G(J^{PC})$
• $\eta_b(1S)$	$0^+(0^{-+})$
• $\Upsilon(1S)$	$0^-(1^{--})$
• $\chi_{b0}(1P)$	$0^+(0^{++})$
• $\chi_{b1}(1P)$	$0^+(1^{++})$
• $h_b(1P)$	$0^-(1^{+-})$
• $\chi_{b2}(1P)$	$0^+(2^{++})$
$\eta_b(2S)$	$0^+(0^{-+})$
• $\Upsilon(2S)$	$0^-(1^{--})$
• $\Upsilon_2(1D)$	$0^-(2^{--})$
was $\Upsilon(1D)$	
• $\chi_{b0}(2P)$	$0^+(0^{++})$
• $\chi_{b1}(2P)$	$0^+(1^{++})$
• $h_b(2P)$	$0^-(1^{+-})$
• $\chi_{b2}(2P)$	$0^+(2^{++})$
• $\Upsilon(3S)$	$0^-(1^{--})$
• $\chi_{b1}(3P)$	$0^+(1^{++})$
• $\chi_{b2}(3P)$	$0^+(2^{++})$
• $\Upsilon(4S)$	$0^-(1^{--})$
aka $\Upsilon(10580)$	
$\Upsilon(10753)$	$?^?(1^{--})$
• $\Upsilon(10860)$	$0^-(1^{--})$
• $\Upsilon(11020)$	$0^-(1^{--})$

meson

Stable or unstable under strong interactions

$\bar{u}u$

- $\pi^\pm$
- π^0
- η
- $f_0(500)$ or σ was $f_0(60$
- $\rho(770)$
- $\omega(782)$
- $\eta'(958)$
- $f_0(980)$
- $a_0(980)$
- $\phi(1020)$
- $h_1(1170)$
- $b_1(1235)$
- $a_1(1260)$
- $f_2(1270)$
- $f_1(1285)$
- $\eta(1295)$
- $\pi(1300)$
- $a_2(1320)$
- $f_0(1370)$
- $h_1(1380)$
- $\pi_1(1400)$
- $\eta(1405)$
- $f_1(1420)$
- $\omega(1420)$
- $f_2(1430)$
- $a_0(1450)$
- $\rho(1450)$

$\bar{s}u$

- $K^\pm$
- K^0
- $K_S{}^0$
- $K_L{}^0$
- $K_0^*(800)$ or
- $K^*(892)$
- $K_1(1270)$
- $K_1(1400)$
- $K^*(1410)$
- $K_0^*(1430)$
- $K_2^*(1430)$
- $K(1460)$
- $K_2(1580)$
- $K(1630)$
- $K_1(1650)$
- $K^*(1680)$
- $K_2(1770)$
- $K_3^*(1780)$
- $K_2(1820)$
- $K(1830)$

$\bar{c}u$

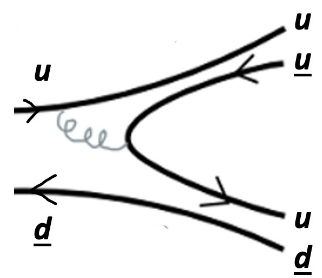
- $D^\pm$
- D^0
- $D^*(2007)^0$
- $D^*(2010)^\pm$
- $D_0^*(2400)^0$
- $D_0^*(2400)^\pm$
- $D_1(2420)^0$
- $D_1(2420)^\pm$
- $D_1(2430)^0$
- $D_2^*(2460)^0$
- $D_2^*(2460)^\pm$
- $D(2550)^0$
- $D(2600)$
- $D^*(2640)^\pm$
- $D(2750)$

❑ strongly stable states

$$\pi^- \to \mu^- \nu_\mu$$

❑ strongly decaying resonances

$$\rho \to \pi\pi$$



Exotic hadrons

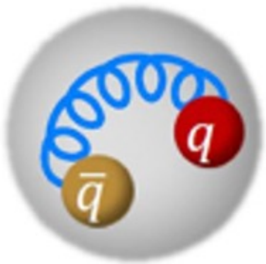
Exotic hadrons

$\pi_1(1600)$

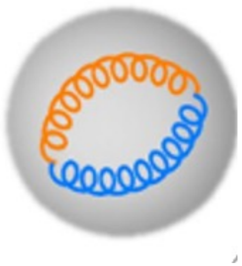
$J^{PC} = 1^{-+}$

PDG (2024)

hybrid meson



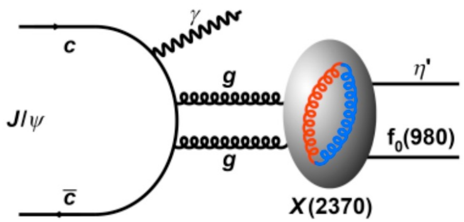
glueball

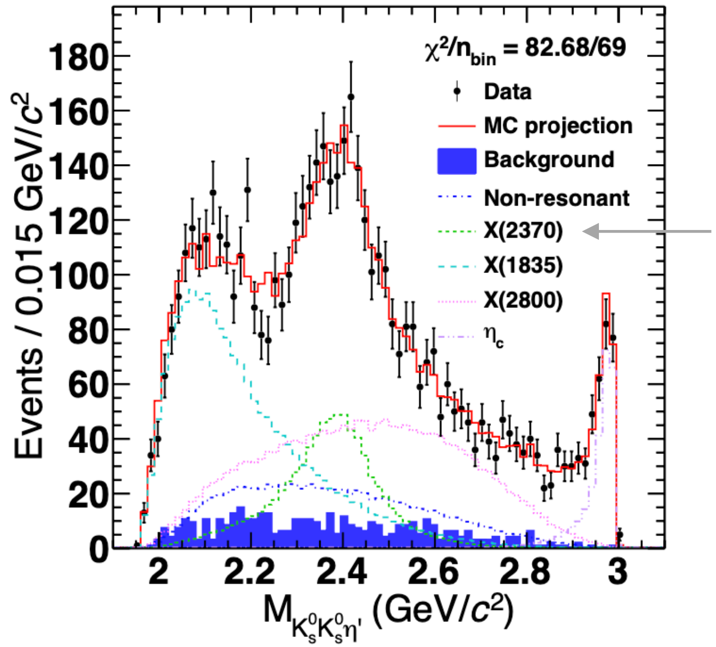


$X(2371)$

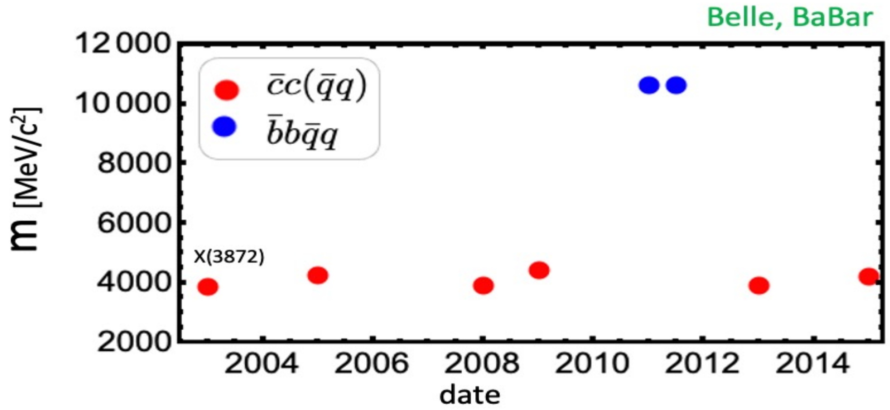
$J^P = 0^-$

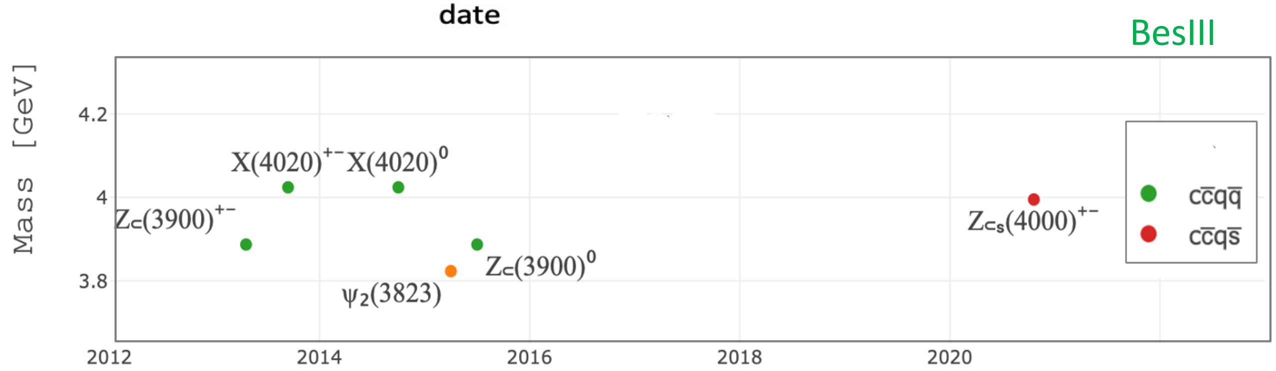
BESIII PRL **132**, 181901 (2024)





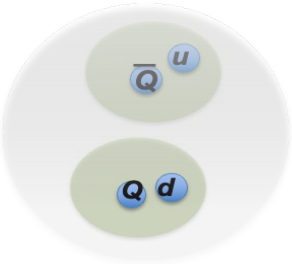
Exotic hadrons



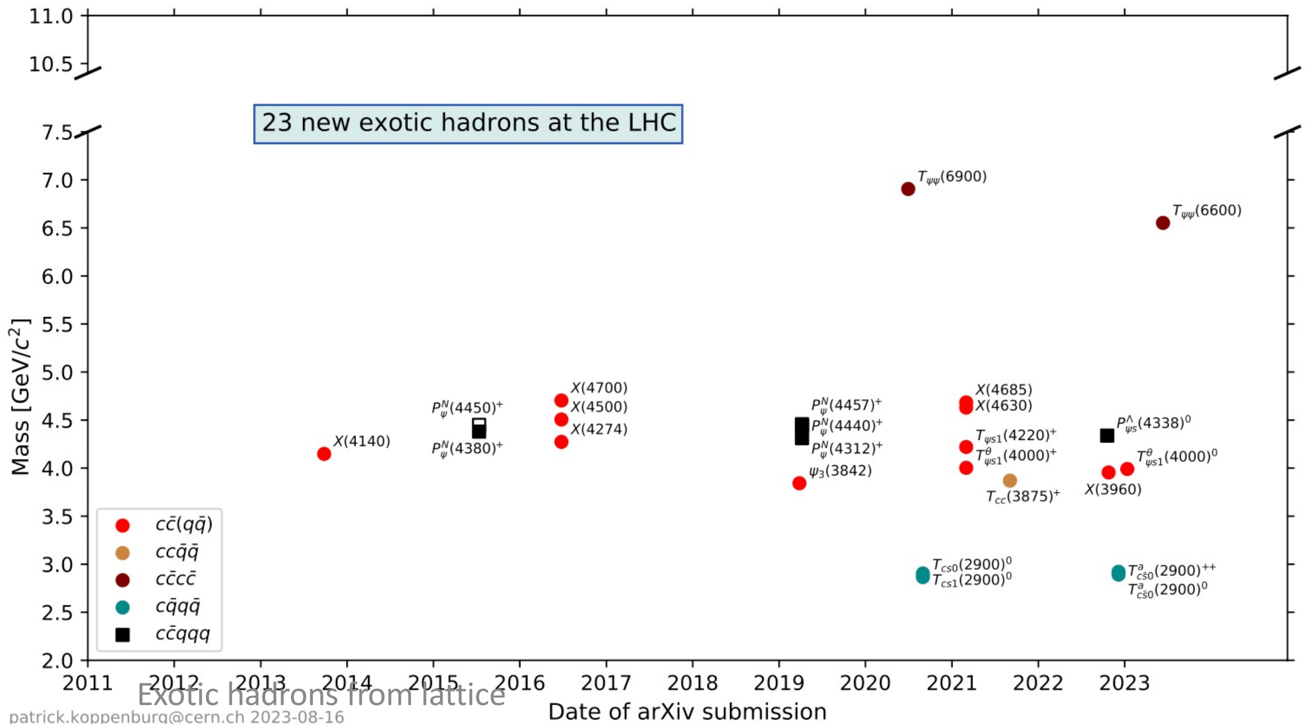


Simplistic argument: for a given V:

heavier particles are easier to bind



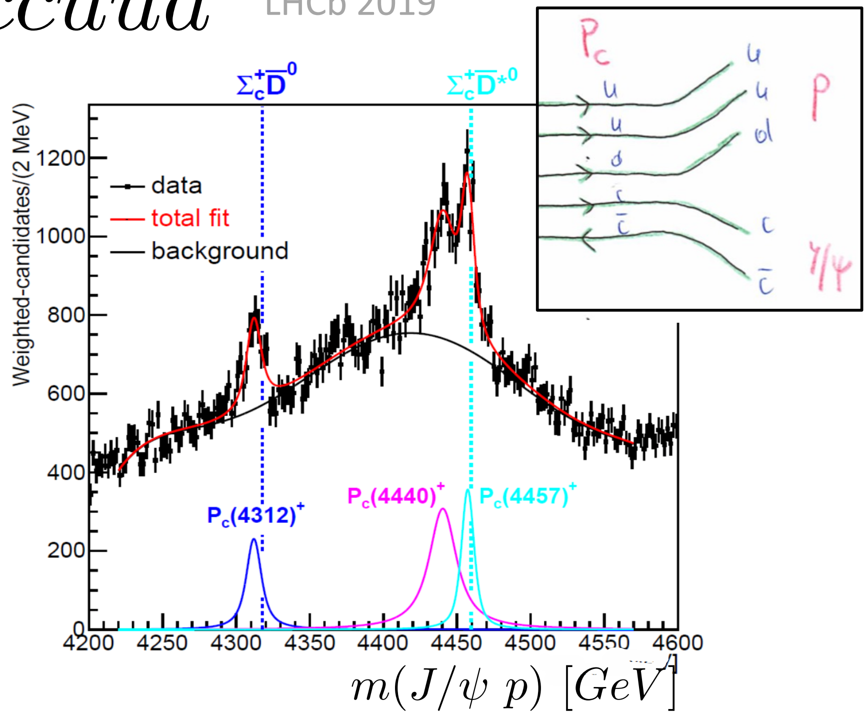
$$\hat{H} = \frac{\hat{p}^2}{2m_r} + V$$



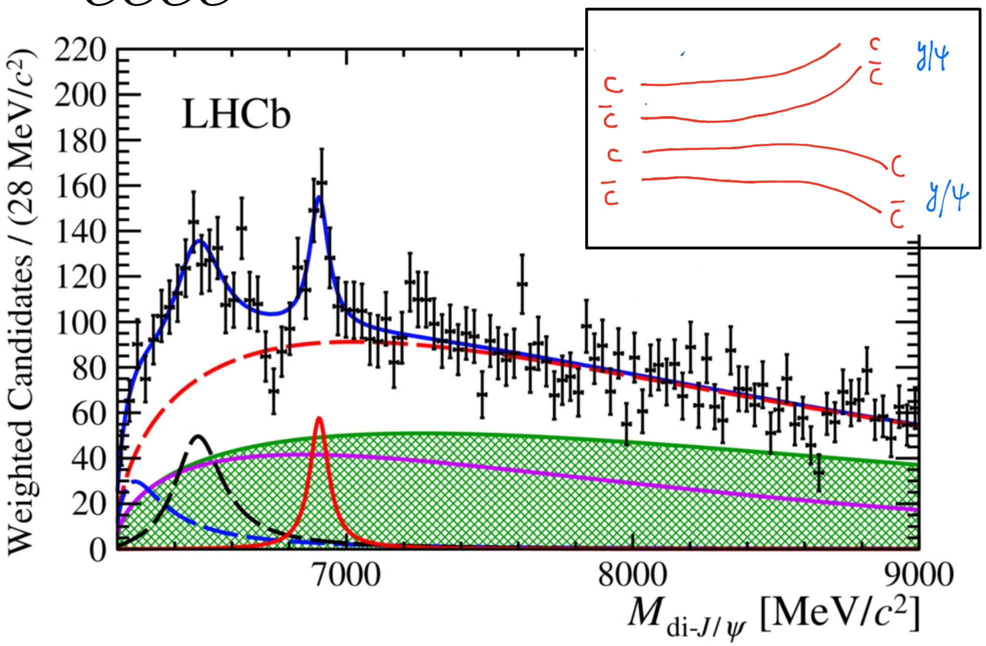
https://www.nikhef.nl/~pkoppenb/particles.html

https://qwg.ph.nat.tum.de/exoticshub/

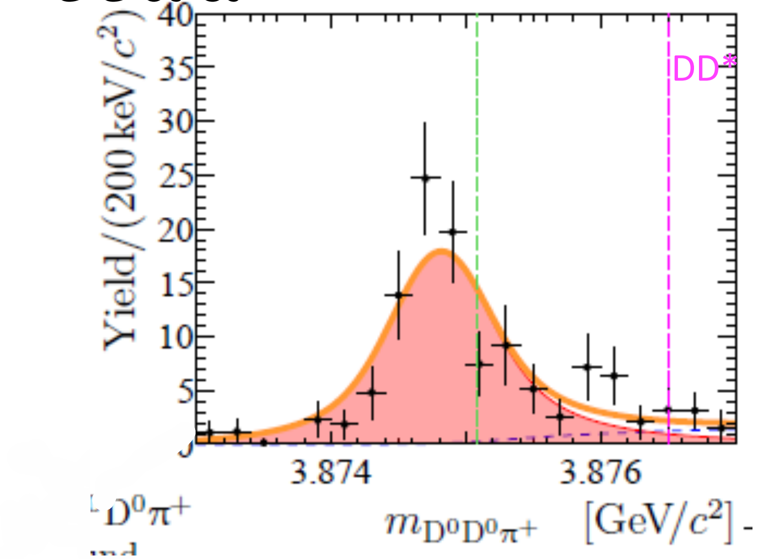
$\bar{c}cuud$ LHCb 2019



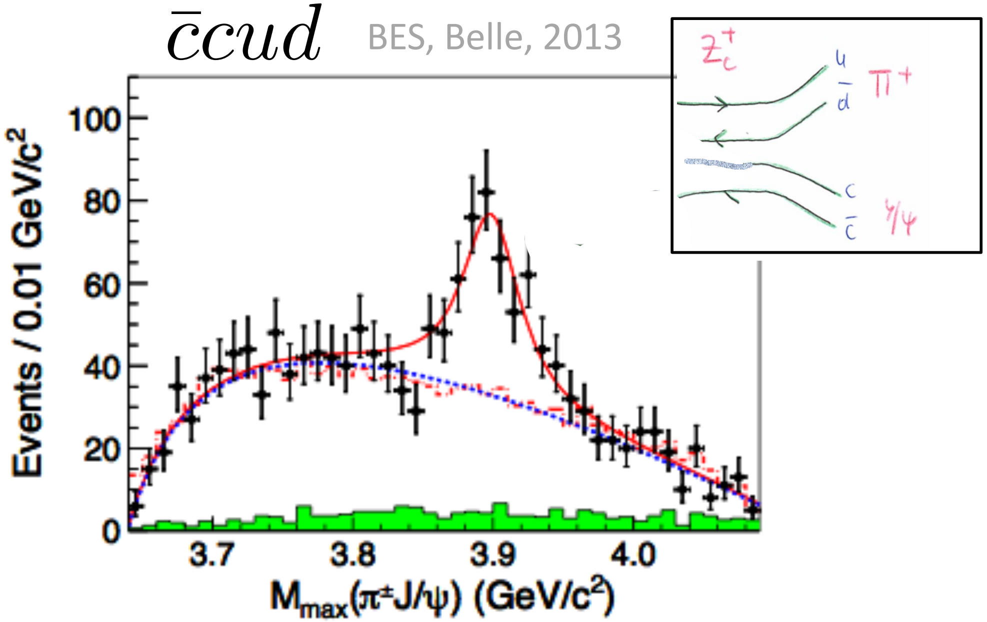
$\bar{c}c\bar{c}c$ LHCb 2021



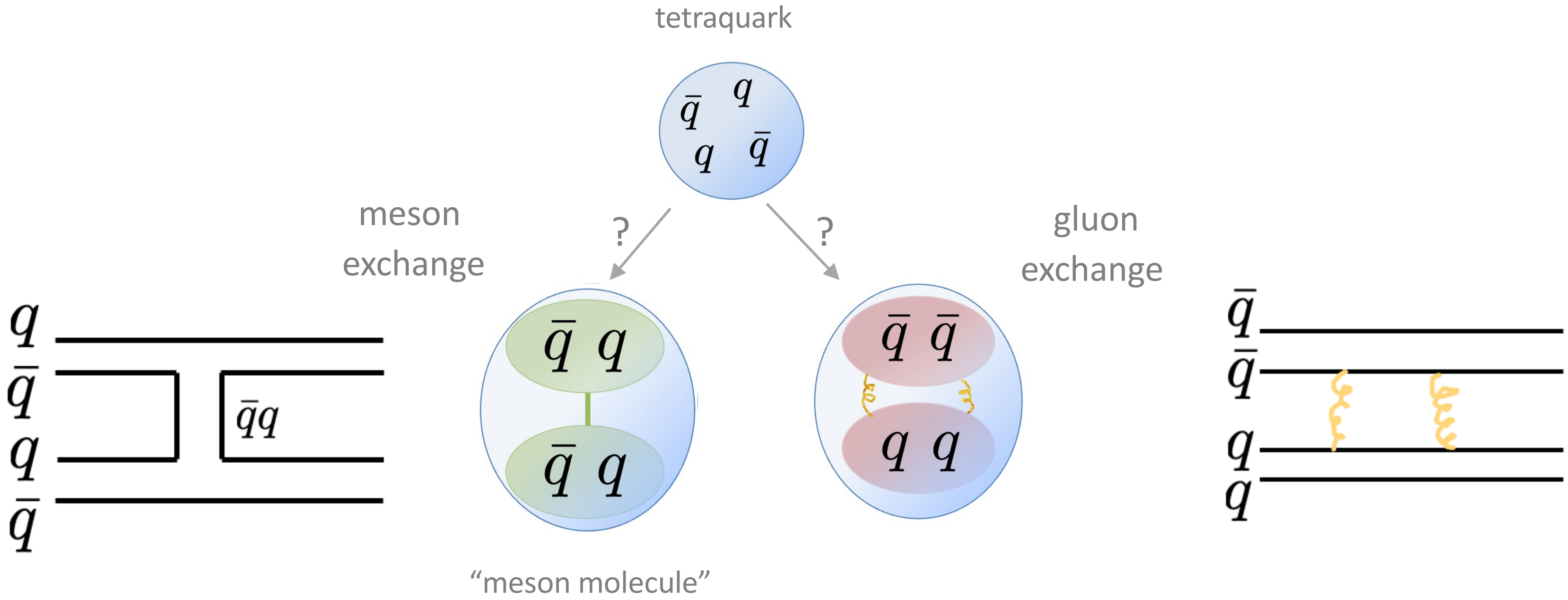
$cc\bar{d}\bar{u}$ LHCb, 29th Jul 2021



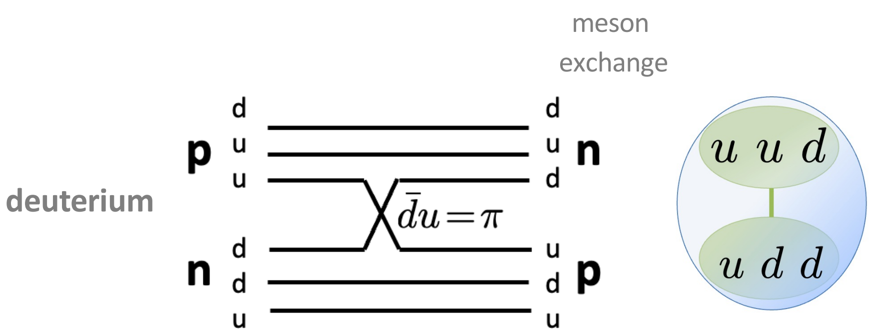
$\bar{c}cu\bar{d}$ BES, Belle, 2013



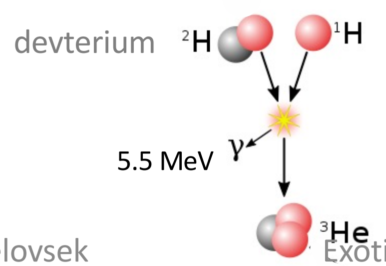
Binding mechanism in exotic hadrons: open problem



Binding mechanism in nuclei



$m_d \approx m_p + m_n$
$- 2.2\ \mathrm{MeV}$



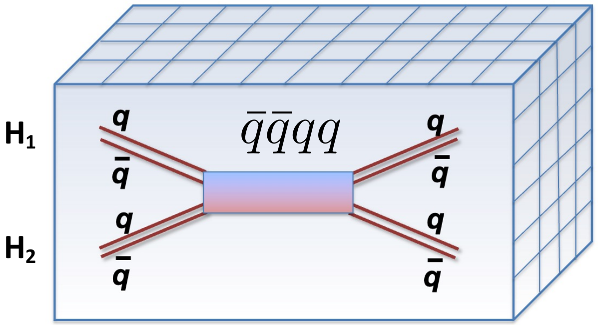
part of fusion chain
in Sun

One of the aims: binding energies

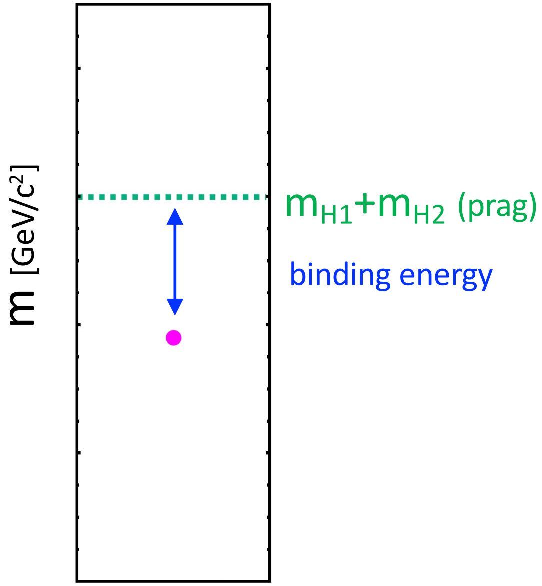
$m \lesssim m_{H1} + m_{H2}$

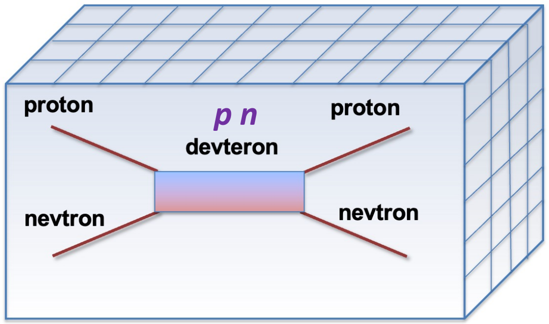
Tetraquark

sipanje

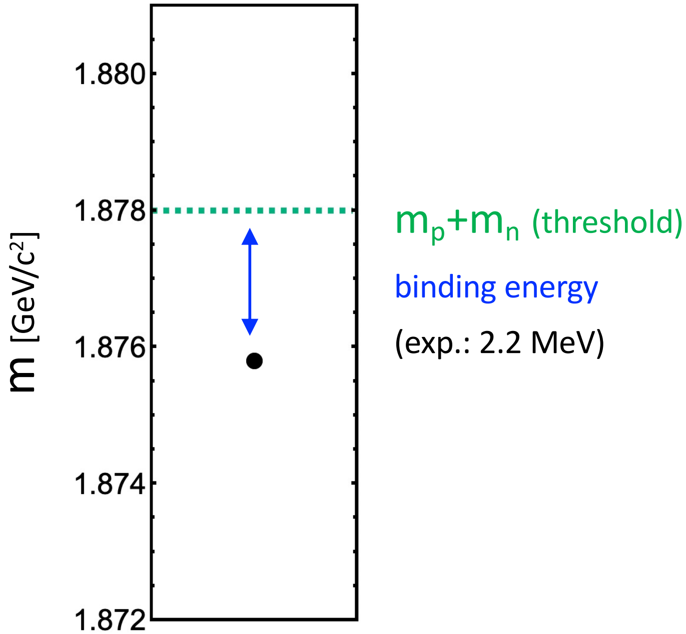


T(E=m) = ∞





$$T(E) \propto \frac{1}{E^2 - m^2 c^4}$$



T(E) = scattering amplitude

m = devterium mass

pole: T(E=m) = ∞

Deuterium

This is not a review

an excellent book on Lattice QCD: Gattringer, Lang

For an overview on spectroscopy : consult papers or review papers

Plenary talk at lattice 2024 in Liverpool:
Hadron Spectroscopy from lattice QCD: current status and future
Nilmani Mathur
https://conference.ippp.dur.ac.uk/event/1265/contributions/7240/attachments/6046/8075/Lattice24_talk_NMathur.pdf

Nilmani Mathur, Tata

Scattering processes and resonances from lattice QCD
R. Briceno, J. Dudek, R. Young
1706.06223, Rev. Mod. Phys

review:
Tetraquarks and pentaquarks in lattice QCD with light and heavy quarks
P. Bicudo
2212.07793, *Phys.Rept.* 1039 (2023)

review:
The *XYZ* states: experimental and theoretical status and perspectives
N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. Thomas, A. Vairo, C.-Z. Yuang
1907.07583, Physics Reports
(there is section on lattice results)

proceedings of Lattice conferences, parallel or plenary talks , ...

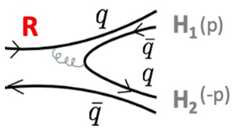
A number of works done in valuable collaboration with Madanagopalan Padmanath (IMSc Chennai)

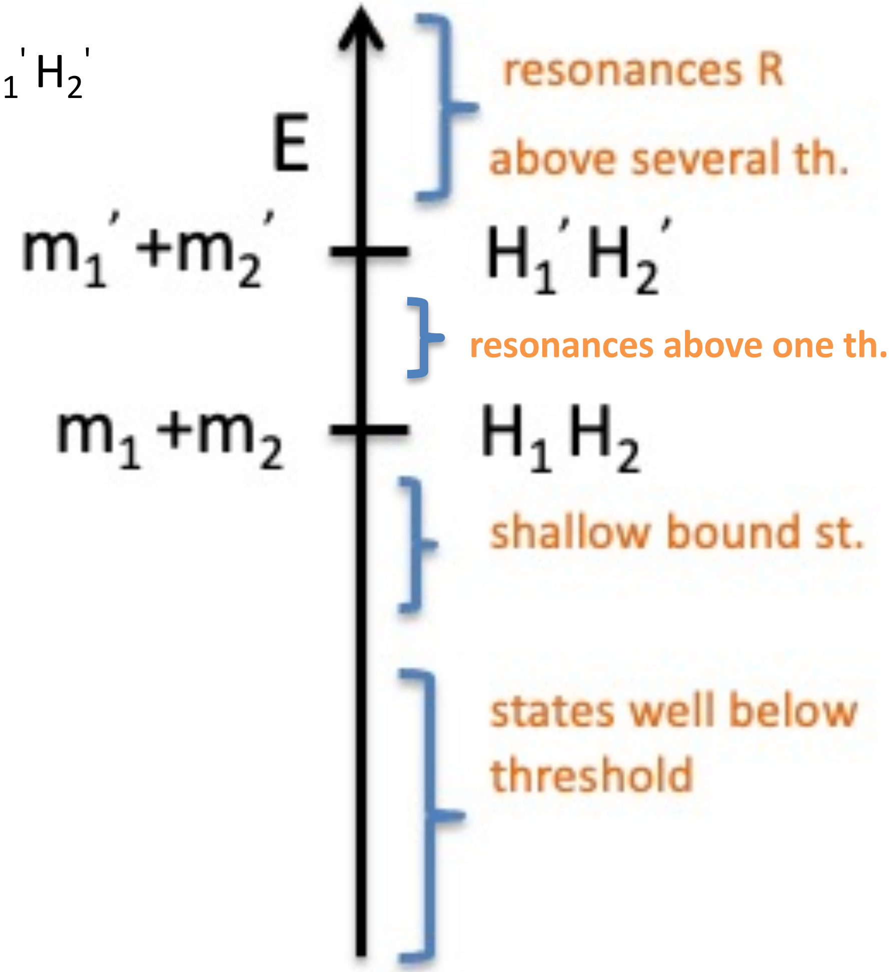
How difficult it is to study a given hadron?

lattice QCD: strong, ~~EW~~

$R \rightarrow H_1 H_2, H_1' H_2'$

$R \rightarrow H_1 H_2$



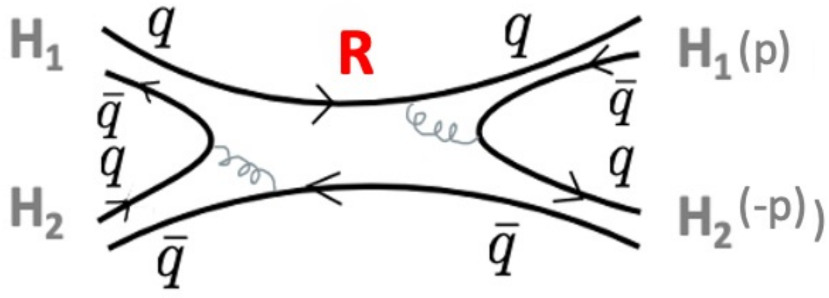


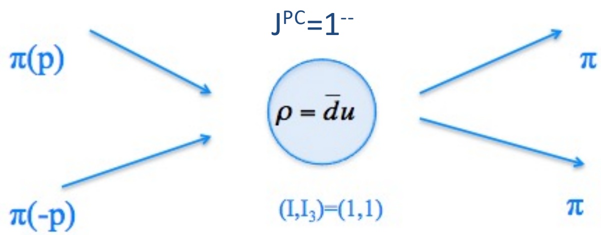
Outline

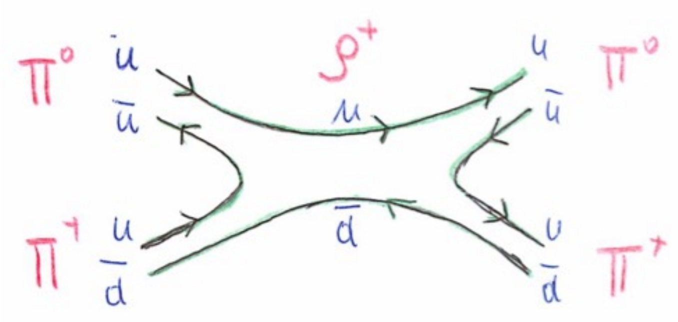
- hadrons from static potentials
- hadrons from coupled-channel scat.
- hadrons from one-channel scattering
- hadrons well below threshold

- **most of examples: exotic hadrons**
- this is NOT a review of all existing results !

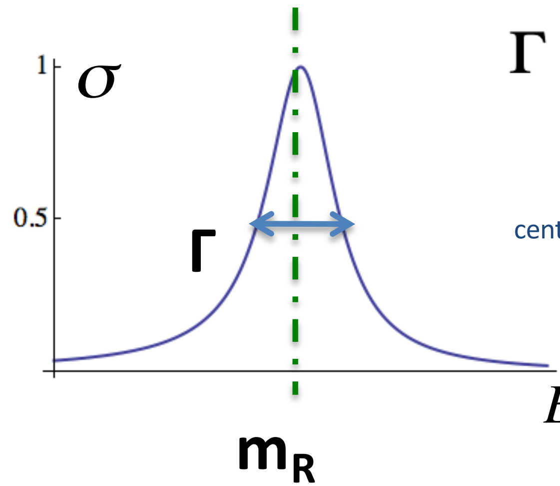
Resonances appear in scattering as “bumps” in cross-section







in experiment and in theory one determines:



$$\Gamma \cdot \tau = \hbar$$

center-of-momentum energy

invariant mass

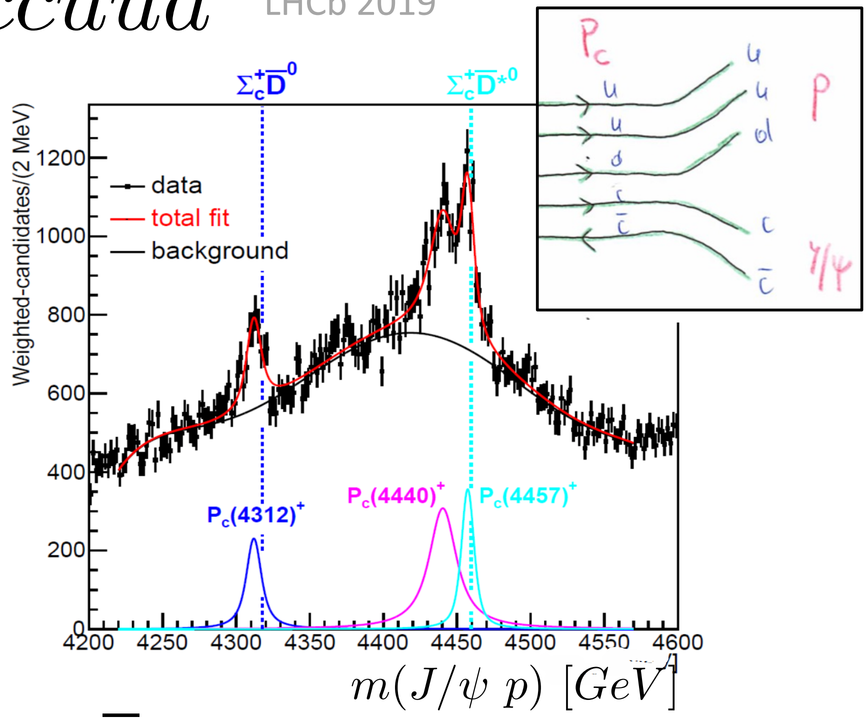
$$E(\pi\pi) = m(\pi\pi)$$

$$\sigma(E) \propto |T(E)|^2 = |\frac{E\Gamma}{E^2 - m_R^2 + iE\Gamma}|^2$$

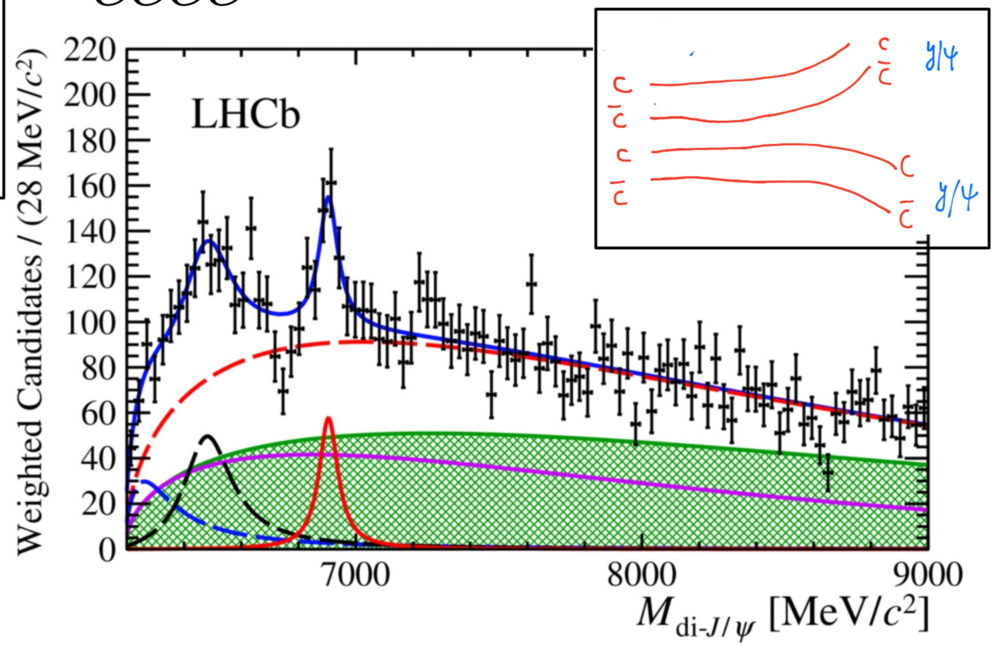
simplest Breit Wigner

scattering amplitude: basic object of these lectures

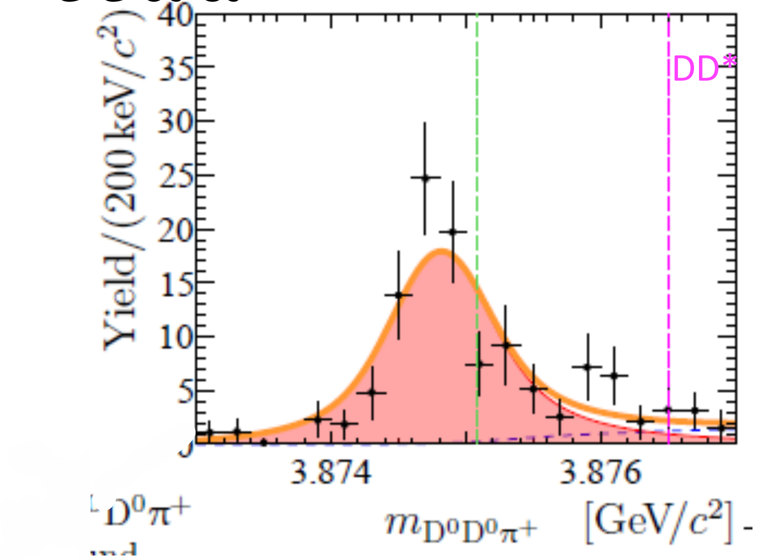
$\bar{c}cuud$ LHCb 2019



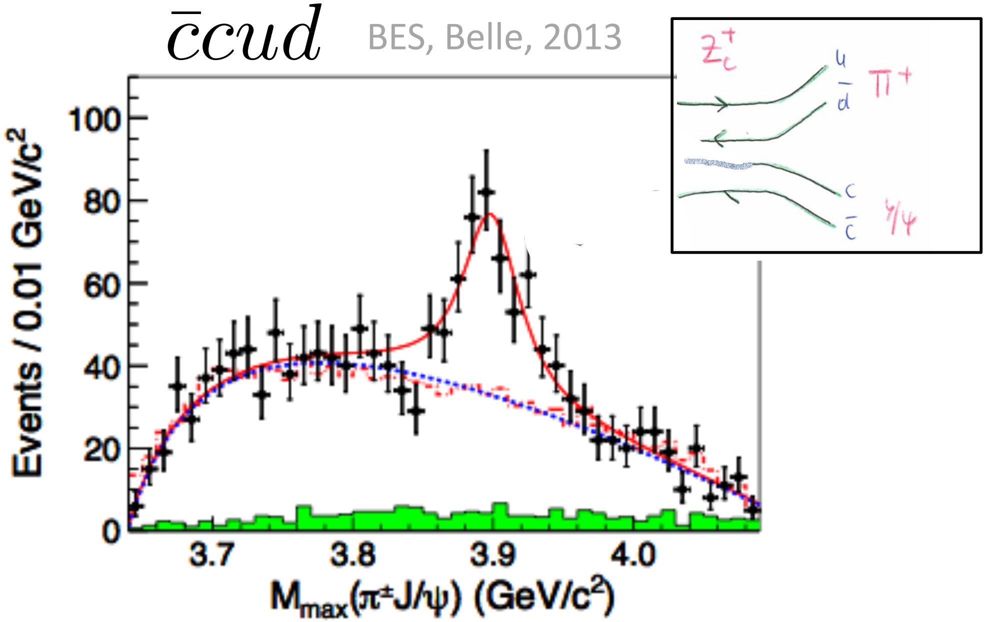
$\bar{c}c\bar{c}c$ LHCb 2021



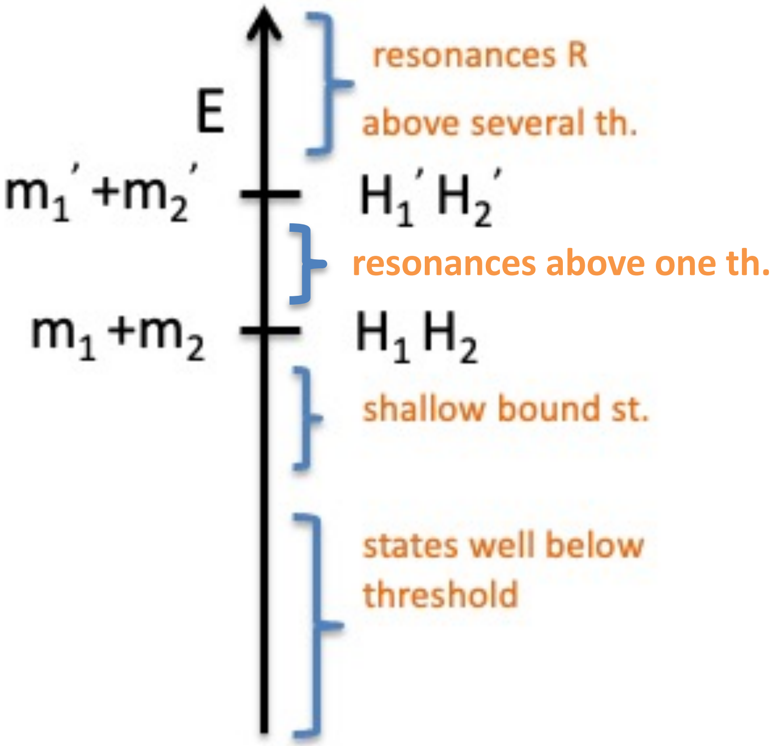
$cc\bar{d}\bar{u}$ LHCb, 29th Jul 2021



$\bar{c}cu\bar{d}$ BES, Belle, 2013



Outline



- hadrons from static potentials
- hadrons from coupled-channel scat.
- hadrons from one-channel scattering

  relation of E_n and scattering amplitude

  scattering in infinite volume
- hadrons well below threshold

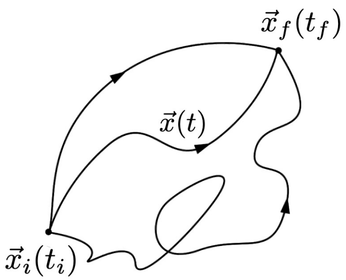
  extracting eigen-energies E_n from lattice

$$\hat{H}_{QCD}|n\rangle = E_n|n\rangle$$

Theoretical challenge: $\mathcal{L}_{QCD} = \frac{1}{4} G_a^{\mu\nu} G_a^{\mu\nu} + \bar{q} i \gamma_\mu (\partial^\mu + i g_s G_a^\mu T^a) q - m_q \bar{q} q$ $g_s \not\ll 1$ at hadronic energy scale

Lattice QCD is based on Feynman path integral

Quantum machanics



$$\langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle \propto \int \mathcal{D}x(t)\ e^{iS/\hbar}$$

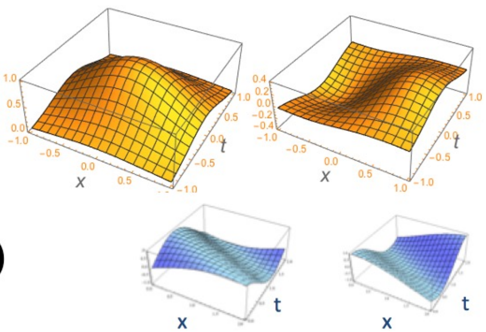
sum over all paths $\vec{x}(t)$

$$S = \int dt L$$
$$= \int dt\ [\tfrac{m}{2} \dot{\vec{x}}^2 - V(\vec{x})]$$

Quantum field theory

$q(t,\vec{x})$

$G(t,\vec{x})$

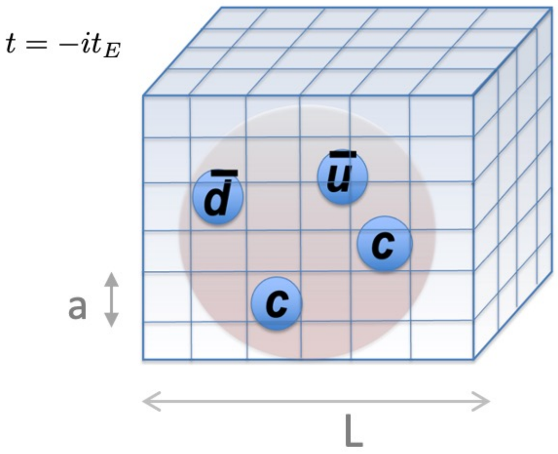


$$\langle C \rangle \propto \int \mathcal{D}G \mathcal{D}q \mathcal{D}\bar{q}\ C\ e^{iS/\hbar}$$

sum over all fields $G(x), q(x)$

$$S = \int d^4x\ \mathcal{L}_{QCD}(q(x), G(x))$$

Quantum ChromoDynamics on lattice



- numerical evaluation of path integral

  in discretized finite Eucledian space-time

  $t_M = -it$

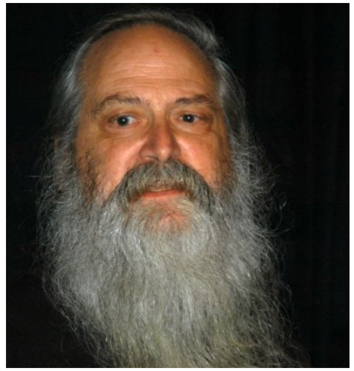
- typical : $a \approx 0.05$ fm , L = 40 a

  a $\to 0$, $L \to \infty$

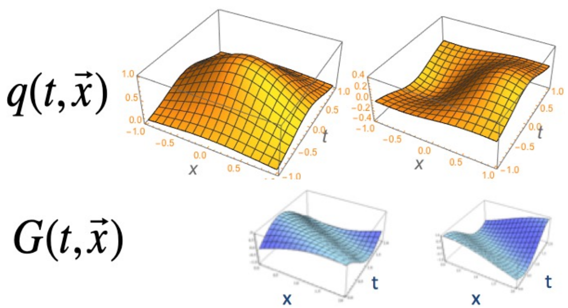
- input: g_s , m_q

first lattice QCD simulation

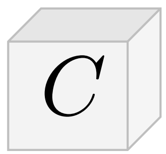
Micahel Creutz : 1980



$$\mathcal{L}_{QCD} = \tfrac{1}{4} G_a^{\mu\nu} G_a^{\mu\nu} + \bar{q} i \gamma_\mu (\partial^\mu + i g_s G_a^\mu T^a) q - m_q \bar{q} q$$



$$\boxed{\langle C \rangle \propto \int \mathcal{D}G \mathcal{D}q \mathcal{D}\bar{q} \; C \; e^{-S_E/\hbar}}$$



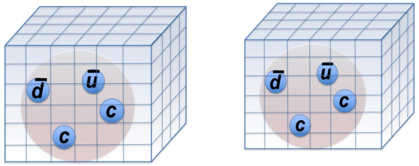
$$S_{QCD} = \int d^4x \; \mathcal{L}_{QCD}$$

Extracting eigen-energies E_n

Main quantity extracted: E_n

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$\sum_n e^{-iE_n t_M}$$



$$C_{ij}(t) = \left\langle 0 \right| \mathcal{O}_i(t)\ \mathcal{O}_j^{+}(0) \left| 0 \right\rangle$$

$$\sum_n |n\rangle\langle n|$$

$$\mathcal{O}(t_M) = e^{iHt_M}\mathcal{O}(0)e^{-iHt_M}$$

$$= \sum_n \langle 0|\ e^{iHt_M}\mathcal{O}_i(0)e^{-iHt_M}\ |n\rangle\langle n|\mathcal{O}_j^{\dagger}(0)|0\rangle$$

$$t_M = -it$$

$$= \sum_n \langle 0|\ e^{Ht}\mathcal{O}_i(0)e^{-Ht}\ |n\rangle\langle n|\mathcal{O}_j^{\dagger}(0)|0\rangle$$

$$= \sum_n \left\langle 0 \middle| \mathcal{O}_i \middle| n \right\rangle\ e^{-E_n t} \left\langle n \middle| \mathcal{O}_j^{+} \middle| 0 \right\rangle$$

$Z_i^n = \langle 0|\mathcal{O}_i|n\rangle$ overlap

$$= \sum_n Z_i^n\ Z_j^{n*}\ e^{-E_n t}$$

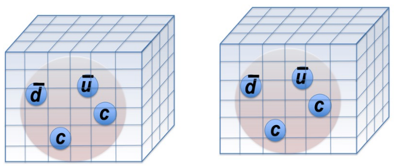
Reminder: All results in this talk will be based on E_n

- for strongly stable state well below threshold : $\boldsymbol{E_n(P=0) \;=\; m}$
- resonances (Luscher's relation) $E_n^{cm} \rightarrow T(E_n^{cm})$
- static potentials: $E_n \rightarrow V(r)$

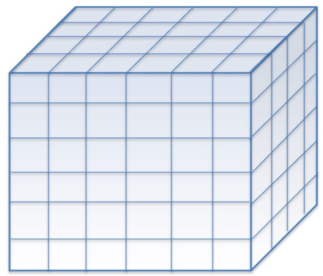
often "non-precision" studies:

single a, $m_{u/d} > m_{u/d}^{phy}$, $m_\pi > 140$ MeV

t, J^P, I t=0, J^P, I



$\mathcal{O} = \mathcal{O}(q, G)$

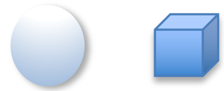


$$C_{ij}(t) = \langle 0 | \mathcal{O}_i(t) \; \mathcal{O}_j^+(0) | 0 \rangle \; = \sum_n \; \langle 0 | \mathcal{O}_i | n \rangle \; e^{-E_n t} \; \langle n | \mathcal{O}_j^+ | 0 \rangle$$

$\vec{p}_1 + \vec{p}_2 = \vec{p}$

Creation/annihilation operators

good quantum numbers: flavor, $\vec{p}$, parity P (for $\vec{p}$=0), $J^P \rightarrow$ irrep



see intermezzo on "Good quantum numbers for reduced rotational symmetry" at the end

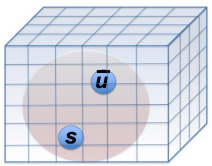
$$\mathcal{O}(x) = \bar{u}(x)\gamma_5 s(x)$$

$$\mathcal{O}_{\vec{p}}(t) = \sum_{\vec{x}} \mathcal{O}(\vec{x},t) e^{i\vec{p}\vec{x}}$$

system with kaon q. n.

$J^P=0^-$

$$(\bar{u}\gamma_5 s)_{\vec{p}}$$
$$(\bar{u}\gamma_t\gamma_5 s)_{\vec{p}}$$
$$(\bar{u}\gamma_t\gamma_5\gamma_i\nabla_i s)_{\vec{p}}$$
$$(\bar{u}\nabla_i\gamma_5\nabla_i s)_{\vec{p}}$$

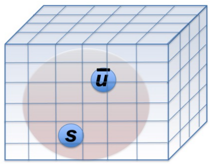


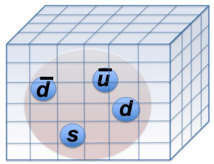
system with K* q. n.

$J^P=1^-$

$$(\bar{u}\gamma_i s)_{\vec{p}}$$
$$(\bar{u}\gamma_t\gamma_i s)_{\vec{p}}$$
$$(\bar{u}\nabla_i s)_{\vec{p}}$$

$$\pi_{\vec{p}_1} K_{\vec{p}_2} = (\bar{u}\gamma_5 d)_{\vec{p}_1} (\bar{d}\gamma_5 s)_{\vec{p}_2} + \{u \rightarrow d\}$$



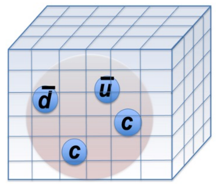


system flavor ccud

$J^P=1^+$

$$D_{\vec{p}_1} D^*_{\vec{p}_2} = (\bar{u}\gamma_5 c)_{\vec{p}_1} (\bar{d}\gamma_i c)_{\vec{p}_2} - \{u \rightarrow d\}$$

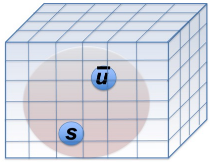
$$(\ [\bar{u}C\gamma_5\bar{d}]_{3_c}\ [cC\gamma_i c]_{\bar{3}_c}\ )_{\vec{p}}$$



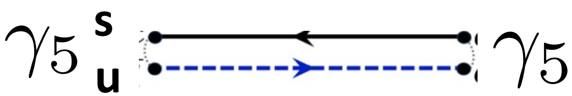
Correlation functions and Wick contractions

system with kaon q. n.

$J^P=0^-$



$$\bar{u}\gamma_5 s$$

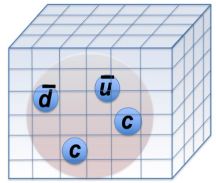


$$\langle C\rangle \propto \int \mathcal{D}G\ [\det(D)]^{N_f} e^{-S_G}$$

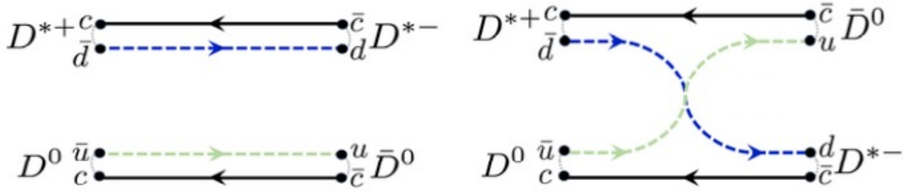
$$\mathrm{Tr}[\gamma_5 D_u^{-1}\gamma_5 D_s^{-1}]$$

system flavor cc<u>ud</u>

$J^P=1^+$



$$D_{\vec{p}_1} D^*_{\vec{p}_2} = (\bar{u}\gamma_5 c)_{\vec{p}_1} (\bar{d}\gamma_i c)_{\vec{p}_2}$$



$$\langle C\rangle \propto \int \mathcal{D}G\ [\det(D)]^{N_f} e^{-S_G}$$

$$(\ \mathrm{Tr}[\Gamma D_c^{-1}\Gamma D_u^{-1}]\mathrm{Tr}[\Gamma D_c^{-1}\Gamma D_d^{-1}]$$

$$-\mathrm{Tr}[\Gamma D_c^{-1}\Gamma D_u^{-1}\Gamma D_c^{-1}\Gamma D_d^{-1}]\ )$$

$$\langle C\rangle \propto \int \mathcal{D}G\mathcal{D}q\mathcal{D}\bar{q}\ C\ e^{-S_G-\bar{q}Dq}$$

$$D_q(G) = i\gamma_\mu(\partial^\mu + ig_s G_a^\mu T^a) - m_q$$

size of D: N x N, $N = N_L^3\ N_T * 4 * 3 \sim 10^8$

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle = \sum_n Z_i^n Z_j^{n*}\; e^{-E_n t} \qquad Z_i^n = \langle 0|\mathcal{O}_i|n\rangle$$

i,j=1,..,N

Extracting E_1

large t: $C_{ij}(t) \propto e^{-E_1 t}$

Extracting E_n and overlaps Z_i^n with GeVP

C. Michael [Nucl. Phys. B 259, 58 1985] , Luscher & Wolf [Nucl. Phys. B 339, 222 1990] , Blossier et al [JHEP 0904, 094 2009]

Recipe:

1) Solve for **λ** and **u** ; t0 is reference time after which only N lowest eigenstates (significantly) contribute

$$C(t)\; u^{(n)}(t) = \lambda^{(n)}(t,t_0)\; C(t_0)\; u^{(n)}(t)$$

eigenstates n = 1,..,N

NxN matrix — number — vector of lenght N

2) Extract E_n from $\boldsymbol{\lambda}^{(n)}$

$$\lambda^{(n)}(t,t_0) = A\; e^{-E_n t}\; [1 + O(e^{-\Delta E t})] \simeq A\; e^{-E_n t}$$

proof on the next slide

3) Extract Z_i^n as

$$Z_i^n = \langle 0|O_i|n\rangle = e^{E_n \frac{t}{2}} \frac{|C_{ik}(t)\, u_k^{(n)}(t)|}{|C^{\frac{1}{2}}(t)\, u^{(n)}(t)|}$$

without proof

$$C(t)\, u^{(n)}(t) = \lambda^{(n)}(t, t_0)\, C(t_0)\, u^{(n)}(t) \longrightarrow \lambda^{(n)} = A\, e^{-E_n t}$$

Simple "proof"

we assume only N states contribute for times between t_0 and t. Then correlation matrix is

$$C_{ij}(t) = \sum_{m=1}^{N} z_i^m z_j^{m*} e^{-E_m t}$$ in relevant time range.

Let's define a vector u^n, which satisfies

$$\sum_{i=1}^{N} u_i^n z_i^{m*} = \delta_{mn}$$ which can always be found

We aim to show that $$\lambda^n(t) = \frac{C(t)\, u^n(t)}{C(t_0)\, u^n(t)}$$ is $$\lambda^{(n)} = A\, e^{-E_n t}$$

$$C(t)\, u^n(t) = \sum_{j=1}^{N} C_{ij}(t)\, u_j^n = \sum_{j=1}^{N} \sum_{m=1}^{N} z_i^m z_j^{m*} e^{-E_m t} u_j^n \quad \delta_{mn}$$

$$= \sum_{m=1}^{N} z_i^m e^{-E_m t} \delta_{mn} = z_i^n e^{-E_n t}$$

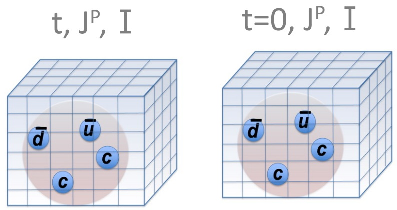
$$\lambda^n(t) = \frac{z_i^n e^{-E_n t}}{z_i^n e^{-E_n t_0}} = e^{-E_n(t-t_0)} = A e^{-E_n t} \quad \square$$

Reminder: All results in this talk will be based on E_n

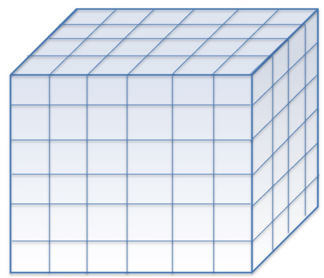
- for strongly stable state well below threshold : $E_n(P=0) = m$
- resonances (Luscher's relation) $E_n^{cm} \rightarrow T(E_n^{cm})$
- static potentials: $E_n \rightarrow V(r)$

often "non-precision" studies:

single a, $m_{u/d} > m_{u/d}^{phy}$, $m_\pi > 140$ MeV

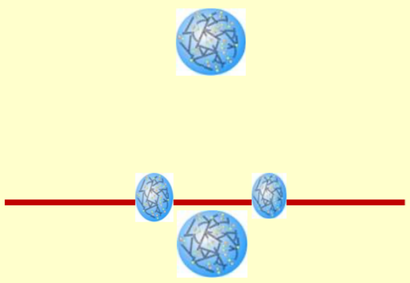
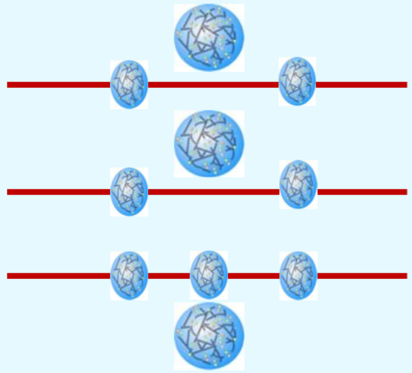


$\mathcal{O} = \mathcal{O}(q, G)$



$$C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^{+}(0) | 0 \rangle = \sum_n \langle 0 | \mathcal{O}_i | n \rangle\, e^{-E_n t} \langle n | \mathcal{O}_j^{+} | 0 \rangle$$

Hadron Spectra: What can Lattice QCD do?

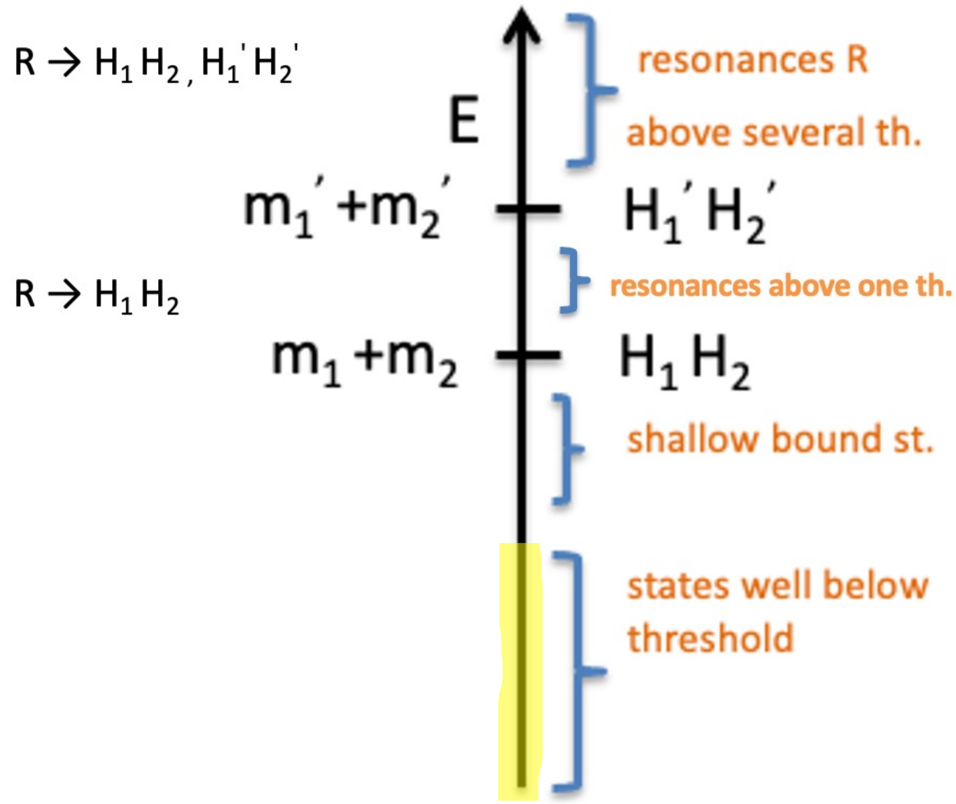
Relatively Easy	Moderately difficult	Difficult
 Stable under strong interactions, Far away below threshold π, K, D, p, n, Λ *etc.* ground states	 • Close to threshold loosely bound, • Elastic resonances with two-body decays $\rho, \Delta, D^*, D^*_{s0}$, deuteron, H-dibaryon, and other hadronic resonances with only 2-body decays	 • Close to multiple thresholds, loosely bound • Inelastic resonances with multiple two-body decays, three or more body X, Y, Z, P_c Pentaquarks, $1^{-+}, \sigma$, *glueballs*, higher nuclei

Power law volume corrections

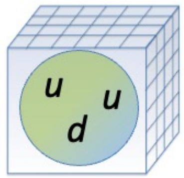
Need rigorous finite volume amplitude analysis with a large basis of operators

16

slide from N. Mathur



$a \to 0, \; L \to \infty, \quad m_q \to m_q^{phy}$

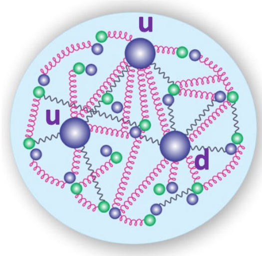


Hadrons well below threshold

(or studied as if located well below threshold)

$$E_n(P=0) \; = \; m$$

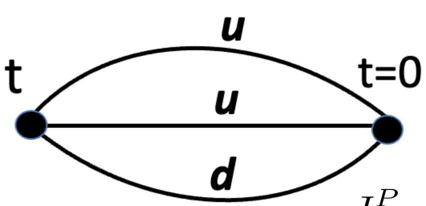
Most important conventional hadron: proton



$$C = \langle 0|\mathcal{O}_p(t) \qquad \mathcal{O}_p^\dagger(0)|0\rangle$$

$$\mathcal{O}_p = \epsilon_{ijk}[u_i^T C\gamma_5 d_j]u_k \simeq uud \qquad J^P = \tfrac{1}{2}^+$$

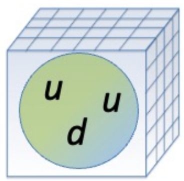
$$C(t) =$$



$$J^P = \tfrac{1}{2}^+$$

$$= \sum_n A_n\, e^{-iE_n t} = \sum_n A_n\, e^{-E_n t_E}$$

$$t = -it_E$$



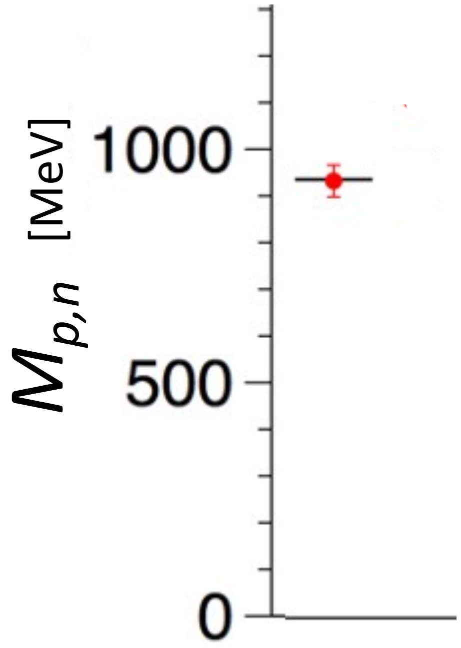
$$\sum_n e^{-iE_n t}$$

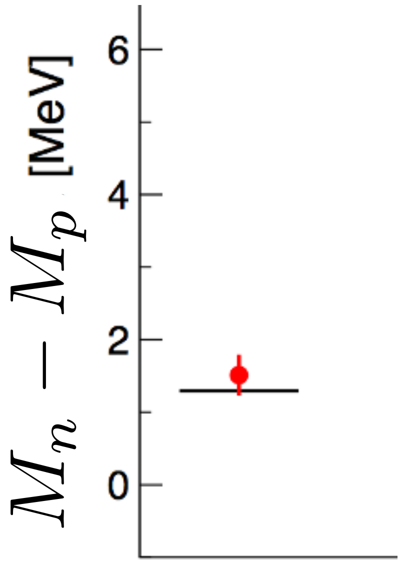
$$\hat{H}_{QCD}\,|n\rangle = E_n|n\rangle$$

$$E^2 = m^2c^4 + \vec{p}^2c^2$$

$$\vec{p} = \vec{0}: \quad E_1 = m_p c^2$$

Proton and neutron mass





— exp

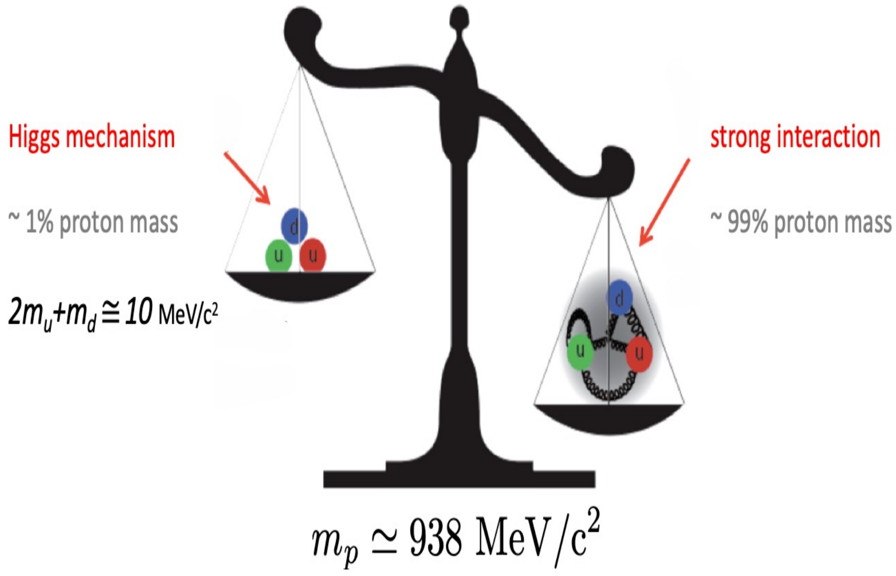
● lattice QCD
$m_u = m_d$

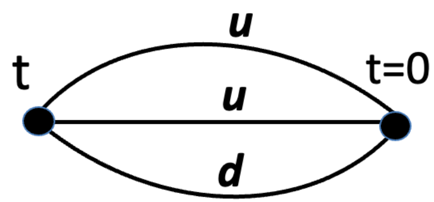
BMW collaboration
Science 322, 2008

— exp

● lattice QCD+QED
$m_u \neq m_d$

BMW collaboration
Science 347,2015





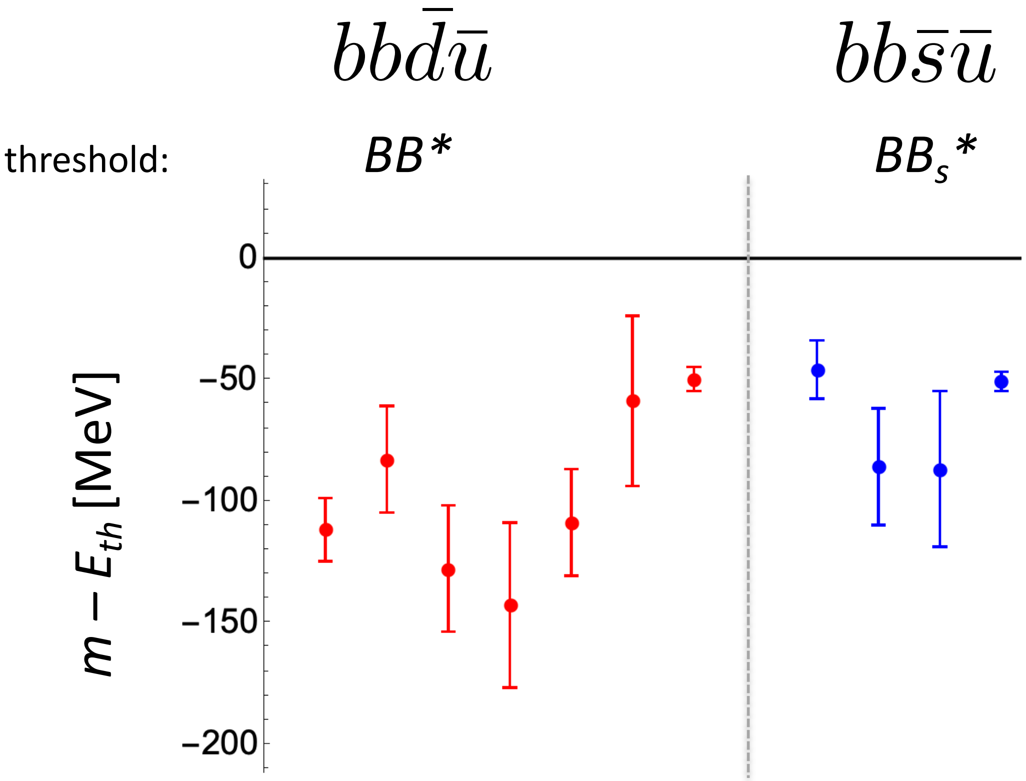
$$C = \langle 0 | \mathcal{O}_p(t)\ \mathcal{O}_p^\dagger(0) | 0 \rangle$$

Doubly bottom tetraquarks

not found in exp, difficult to find

$I=0,\ J^P=1^+$

$bb\bar{d}\bar{u}$ $bb\bar{s}\bar{u}$



$$O = (\bar{u}\gamma_5 b)\ (\bar{d}\gamma_i b) + .. = BB^*$$

$$[b\Gamma_1 b]_{\bar{3}_c} [\bar{u}\Gamma_2 \bar{d}]_{3_c}$$

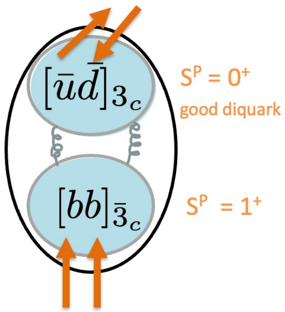
$$\ldots$$

likely dominant

(B and B* to close

in BB* molecule

with binding ~0.1 GeV)



from left to right (lattice QCD)

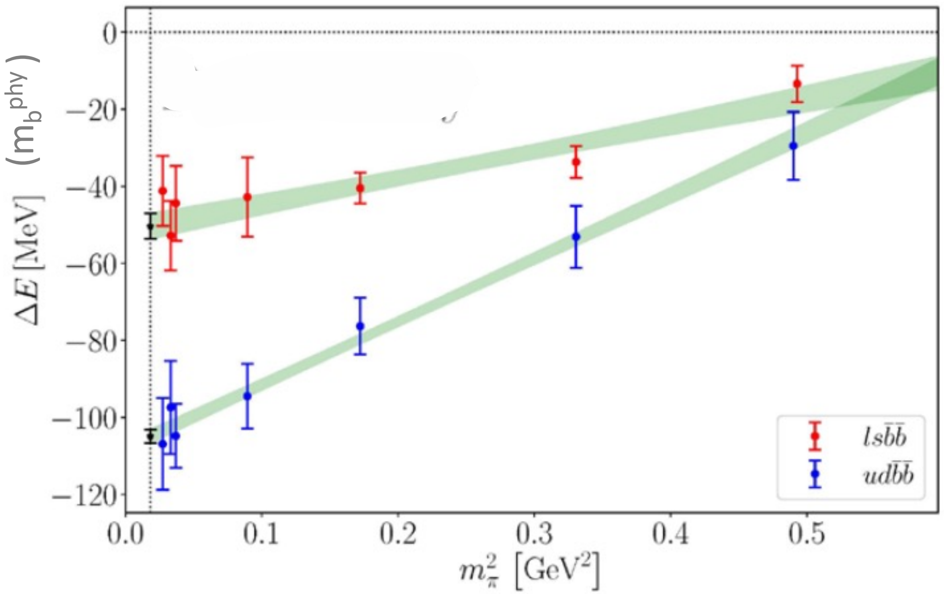
Hudspith, Mohler, 2303.17295
HALQCD, 2306.03565 (cosidering coupling with B*B*)
Leskovec, Meinel, Pflaumer, Wagner, 1904.04197
Junnarkar, Mathur, Padmanth, 1810.12285
Frances, Colquhoun, Hudspith, Maltman (2021 PosLat)
Bicudo, Wagner et al. 1612.02758, static potentials
Brown, Orginost, 1210.1953, static potentials

Hudspith, Mohler, 2303.17295
Meinel, Pflaumer, Wagner, 2205.13982
Junnarkar, Mathur, Padmanth 1810.12285
Frances, Colquhoun, Hudspith, Maltman (2021, PosLat)

Doubly bottom tetraquarks

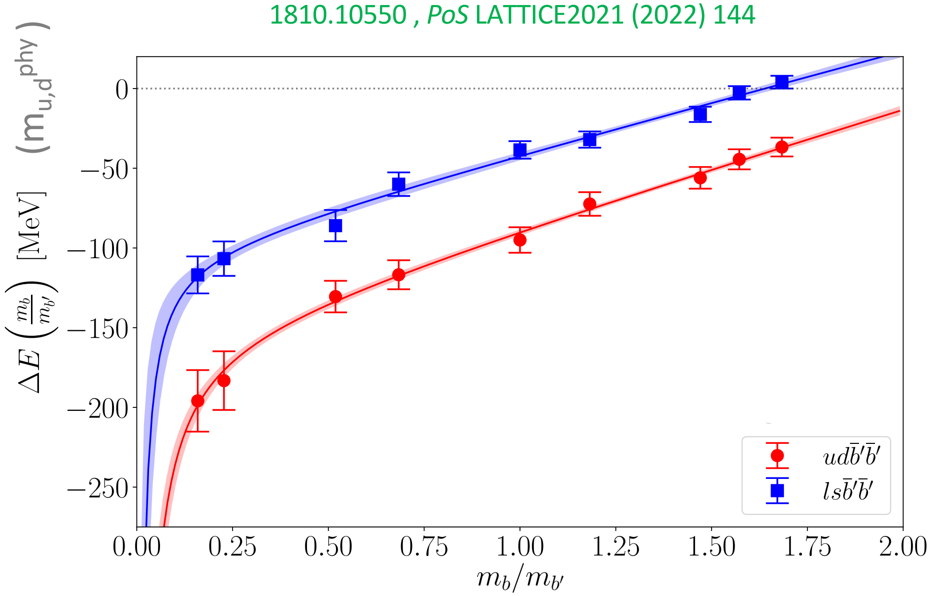
$bb\bar{d}\bar{u}$ $bb\bar{s}\bar{u}$ $I=0,\ J^P=1^+$

lattice: dependence on m_b and $m_{u,d}$



$m_{u,d}$ increases

Colquhoun, Francis, Hudspith, Maltman, Lewis
1810.10550 , *PoS* LATTICE2021 (2022) 144



$m_{b'}$ decreases

Other $QQ'\bar{q}\bar{q}'$ and J^P : $bc\bar{q}\bar{q}'$, $cc\bar{q}\bar{q}'$

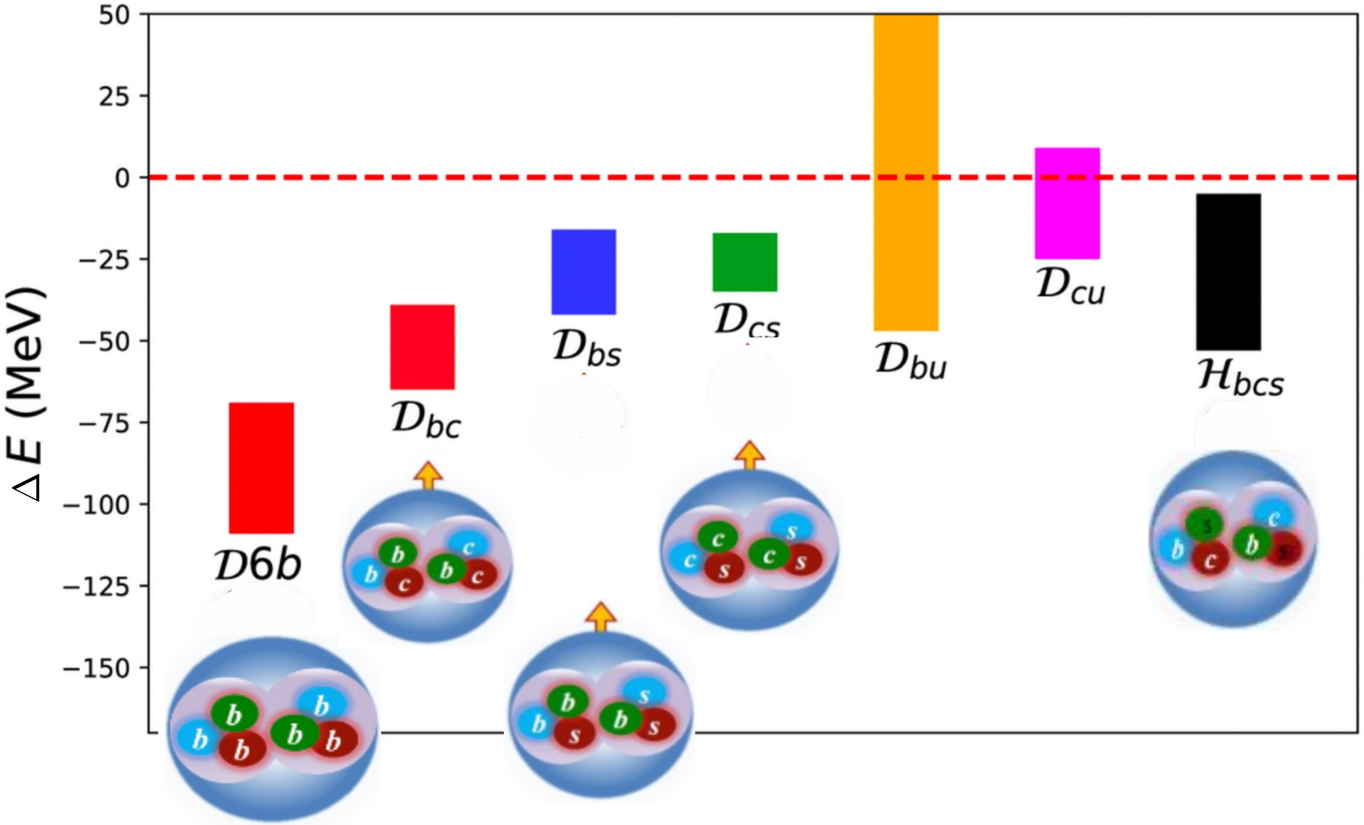
Theoretically expected near or above threshold

States near or above threshold have to be identified from scattering T(E): next Section

Di-baryons with heavy quarks

$$O = qqq\ qqq$$

binding energy $\Delta E = m - m_{B1} - m_{B2}$



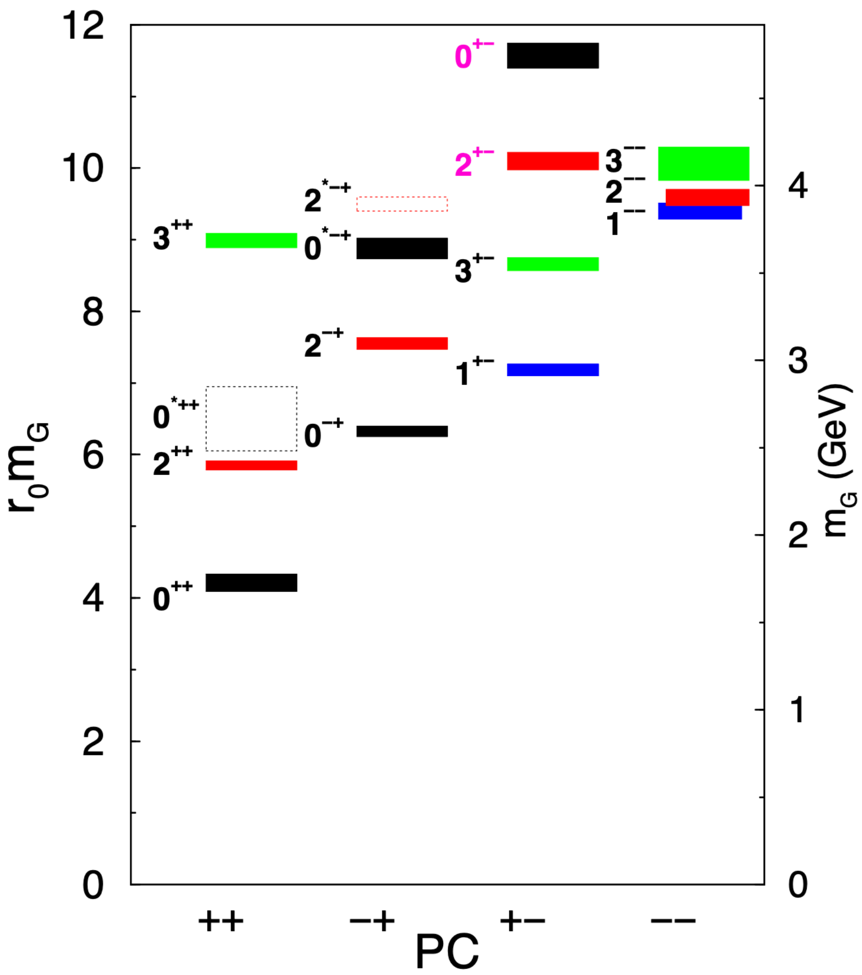
Junnarkar Mathur
1906.06054, PRL

Mathur, Padmanath, Chakraborty
2205.02862

Junnarkar, Mathur, 2206.02942, PRD

Gluebals (no dynamical quarks)

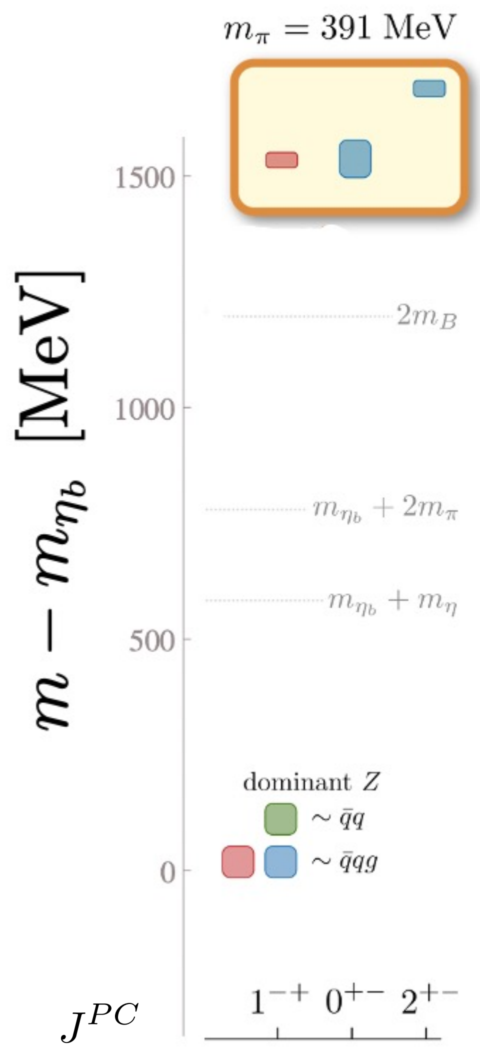
$$GG.. \not\to (\bar{q}q)(\bar{q}q), \ldots$$



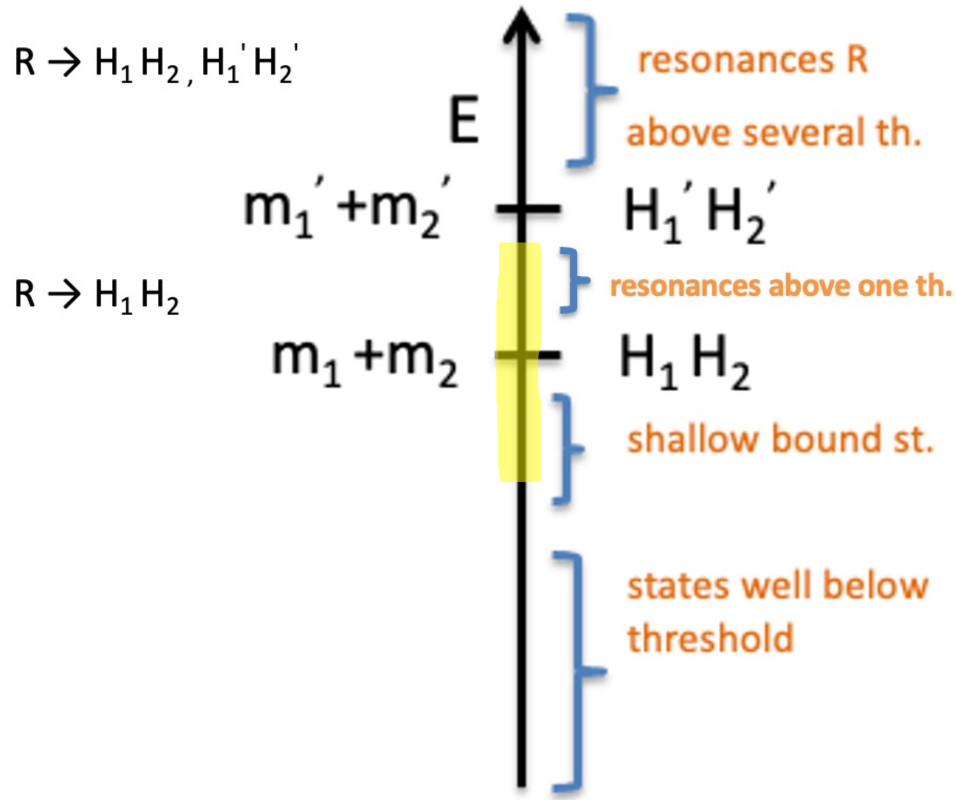
Morningstar & Peardon 1999

Hybrids (omitting strong decays)

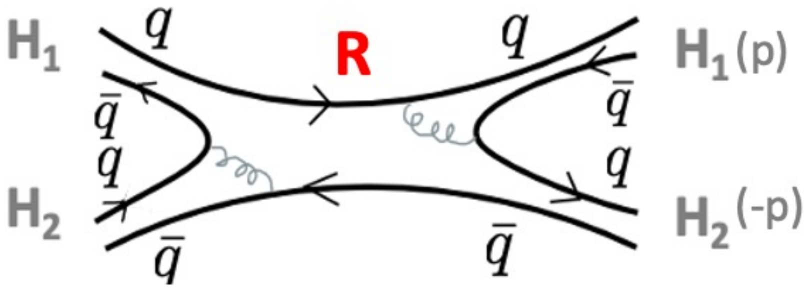
$$\bar{b}Gb$$



Ryan & Wilson (HadSpec) 2008.02656, JHEP



Hadrons from one-channel scattering



Scattering in Nonrelativistic QM, l=0

Schwabl, chapter 18.3

dominant at small E

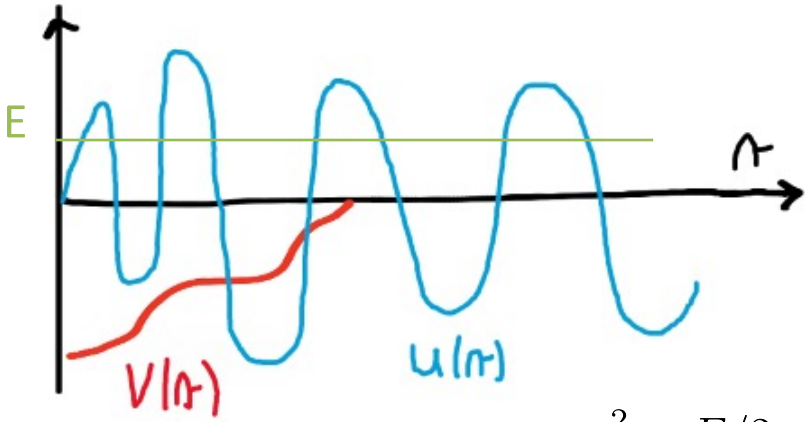
$$\psi(x) \propto e^{i\vec{p}\cdot\vec{x}} + \frac{f}{r} e^{ipr}$$

in (e^{-ipr}) only in the first term

if λ >> object (low p) : l=0 dominant

outside the region of V

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \underbrace{\frac{\hbar^2 l(l+1)}{2mr^2}}_{0} + V(r)\right] u(r) = E u(r)$$



$$p^2 = E/2m$$

u: complicated $\quad u(r) = A\ \sin(pr + \delta_0)$

$$u(r) \propto \sin(pr+\delta_0) = \frac{1}{2i}[e^{ipr+i\delta_0} + e^{-ipr-i\delta_0}]$$

$$= \frac{e^{-i\delta_0}}{2i}[\underbrace{e^{+2i\delta_0}}_{S(E)}\ e^{ipr} + e^{-ipr}]$$

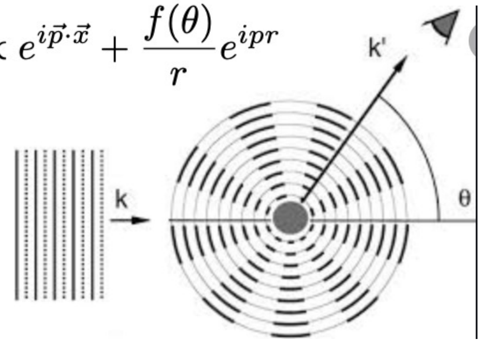
S=<out|in>

$$S(E) = e^{2i\delta_0(E)}\ , \qquad S^\dagger S = I$$

conservation of prob.

Scattering in Nonrelativistic QM, general 1

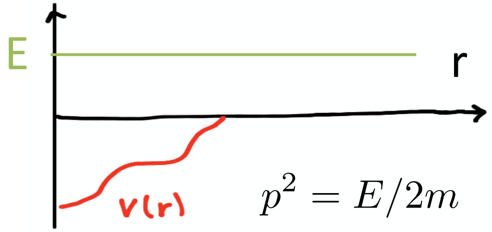
$$\psi(x) \propto e^{i\vec{p}\cdot\vec{x}} + \frac{f(\theta)}{r} e^{ipr}$$



$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right] u(r) = E u(r) \qquad \psi \propto Y_{lm}\frac{u(r)}{r}$$

Homework: try to solve an example with Mathematica **NDSolve**

outside the region of V



$$p^2 = E/2m$$

ingoing wave   outgoing wave

$$u_l(r) \propto \tfrac{r}{2}\,[h_l^{(2)}(pr) + e^{2i\delta_l}\, h_l^{(1)}(pr)]$$

$$u_l(r) \propto \sin(pr - \tfrac{1}{2}\pi l + \delta_l)$$ for large r

T= scattering amplitude: basic object of these lectures

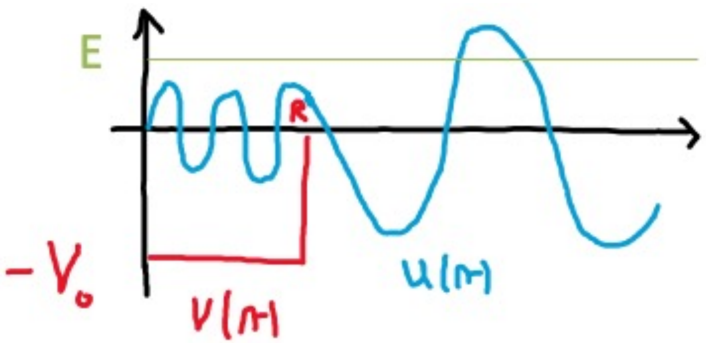
Homework: show that all thee expr. are equivalent

(different normalizations of T are used)

$$S_l(E) = e^{2i\delta_l(E)} = 1 + 2ip\, T_l(E) \;\rightarrow\; T_l = \frac{e^{2i\delta_l} - 1}{2i}\frac{1}{p} = e^{i\delta_l}\sin\delta_l\,\frac{1}{p} = \frac{1}{\cot\delta_l - i}\,\frac{1}{p}$$

$$\boxed{T_l = \frac{1}{p\cot\delta_l - ip}} \qquad \frac{d\sigma}{d\Omega} = \sum_l |T_l|^2$$

Example: phase shift for spherical well and l=0



E > 0 : scattering

relations apply also for E < 0

p: momontum of a scattering particle in cmf

u : $A\,\sin qr$ $\quad B\,\sin(pr+\delta_0)$

$$q^2 = \frac{E+V_0}{2m_r} \qquad p^2 = \frac{E}{2m_r}$$

$$\left.\begin{aligned} u(R) &= A\sin qR = B\,\sin(pR+\delta) \\ u'(R) &= q\;A\cos qR = p\;B\,\cos(pR+\delta) \end{aligned}\right\}$$

dividing both eqs $\quad \frac{1}{q}\tan qR = \frac{1}{p}\tan(pR+\delta)$

$$\boxed{\delta_0(p) = \arctan\left(\frac{p}{q}\tan(qR)\right) - pR + n\pi}$$

$$\delta_0(E, V_0, R)$$

Bound state and virtual state in spherical-well potential

l=0

pn nn

$$\infty$$

$$e^{+2i\delta_0}\ e^{ipr}+e^{-ipr}$$

$$T \propto \frac{1}{p\cot\delta - ip}$$

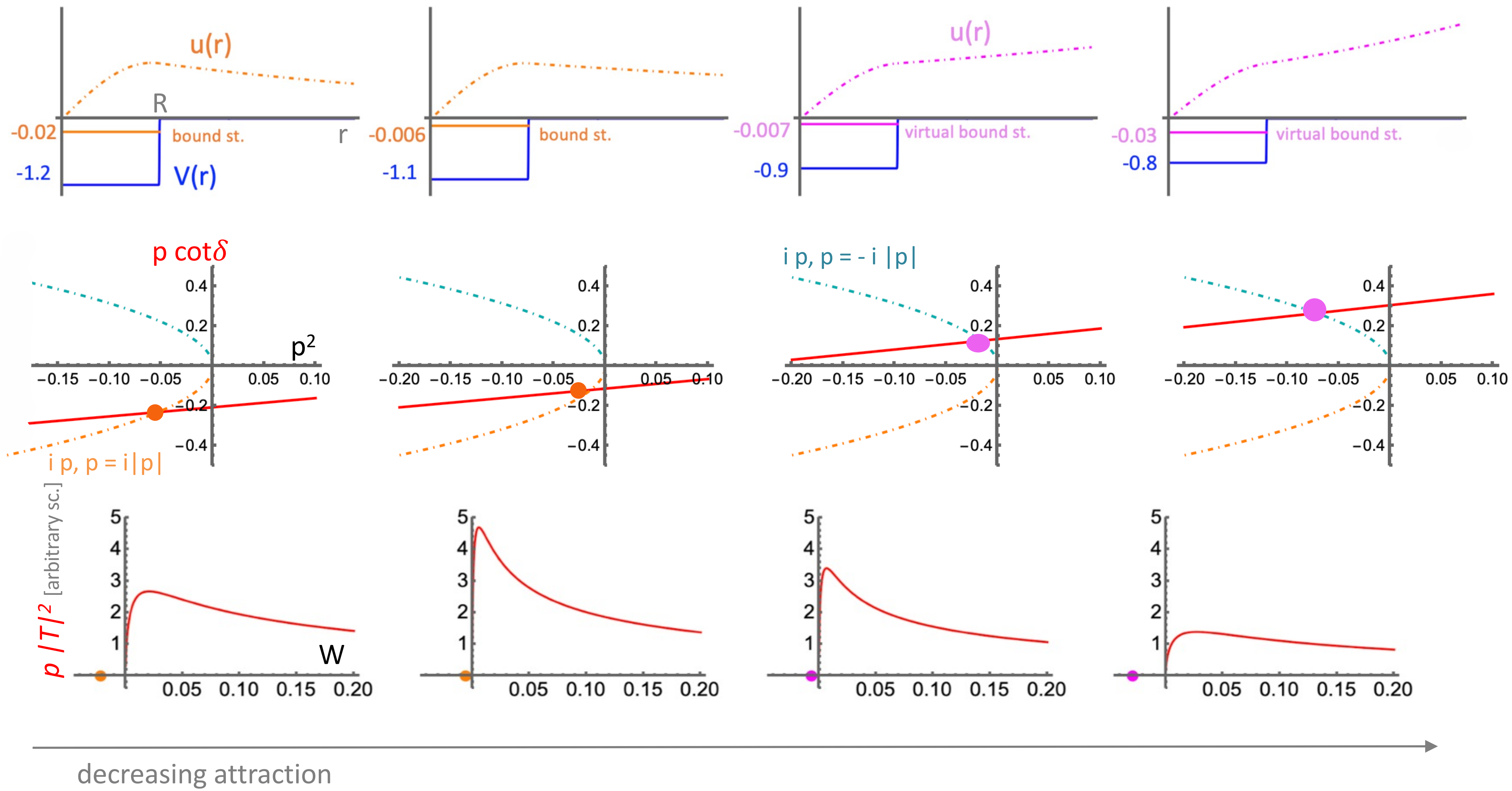
$$u(r) = A\sin(qr) \qquad u(r) = B\sin(pr+\delta)$$

p = i |p|

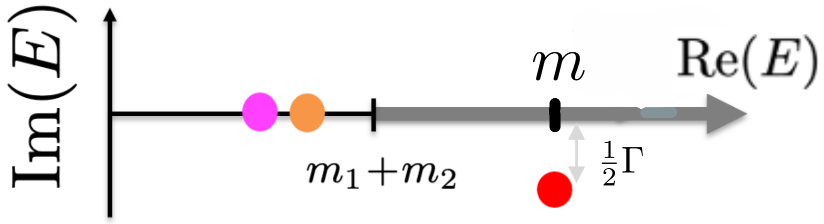
$$e^{ipr} = e^{-|p|r}$$

p = - i |p|

$$e^{ipr} = e^{|p|r}$$



$E = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}$



pole locations in T(E)

in complex energy plane

$E = E^{NR} + m_1 + m_2$

Virtual bound st.

p = - i |p|, sheet II

Bound st.

p = i |p|, sheet I

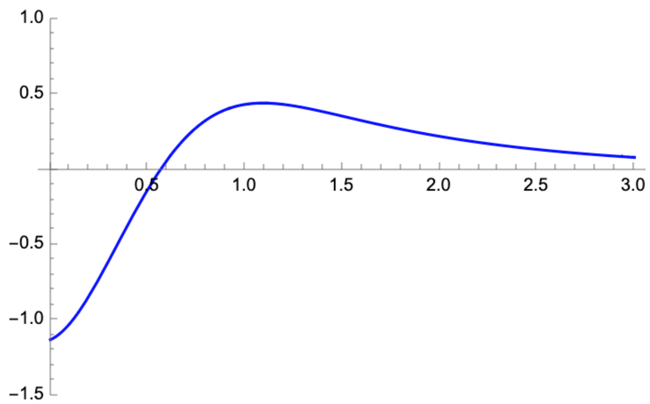
Resonance

sheet II

Resonance in quantum mechanical scattering

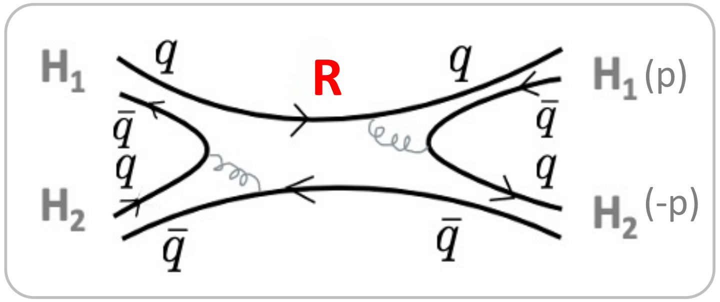
no resonance for fully attractive potential for l=0

V(r)



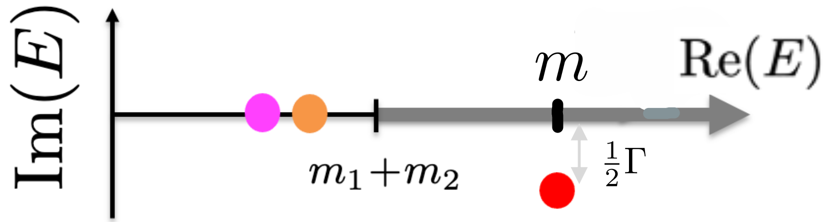
States from one-channel scattering

scattering amplitude T(E)



$$S(E) = e^{2i\delta(E)} = 1 + 2i\,\frac{2p}{E}\,T(E) \rightarrow \boxed{T \propto \frac{1}{p\cot\delta - ip}}$$

E= $\sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}$

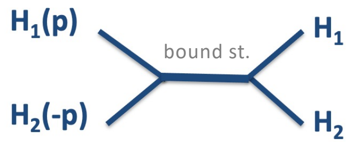


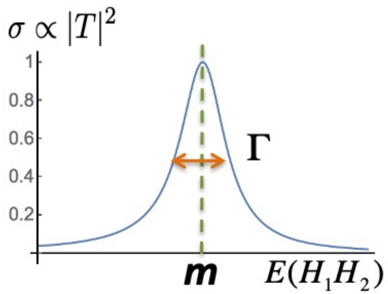
Virtual bound st.	Bound st.	Resonance
p = - i |p|, sheet II	p = i |p|, sheet I	sheet II

$p^2<0$

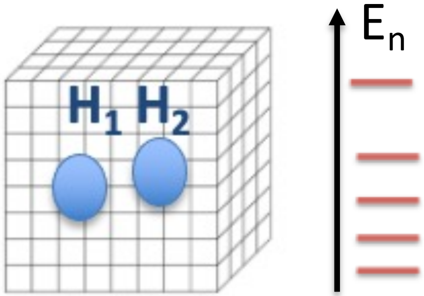
$$T(E) \propto \frac{1}{E^2 - m^2}$$

$$T(E) \propto \frac{1}{E^2 - m^2 + iE\Gamma}$$





Scattering amplitude T(E) from lattice QCD



$$\sigma(E) \propto |T(E)|^2$$

real E — for real E — for complex E

$$E \rightarrow T(E) \rightarrow T(E^c)$$

analytic relation:
Luscher 1991
generalizations by many authors

analytic contin.
to complex E

Relation between E and $\delta(E)$, $T(E)$

[Luscher 1991]

Relation between E and δ(E) , T(E): 1D quantum mechanics

$$S(E) = e^{2i\delta(E)} = 1 + 2i\,\frac{2p}{E}\,T(E)$$

$$E \rightarrow \delta(E), T(E)$$

derivation of relation

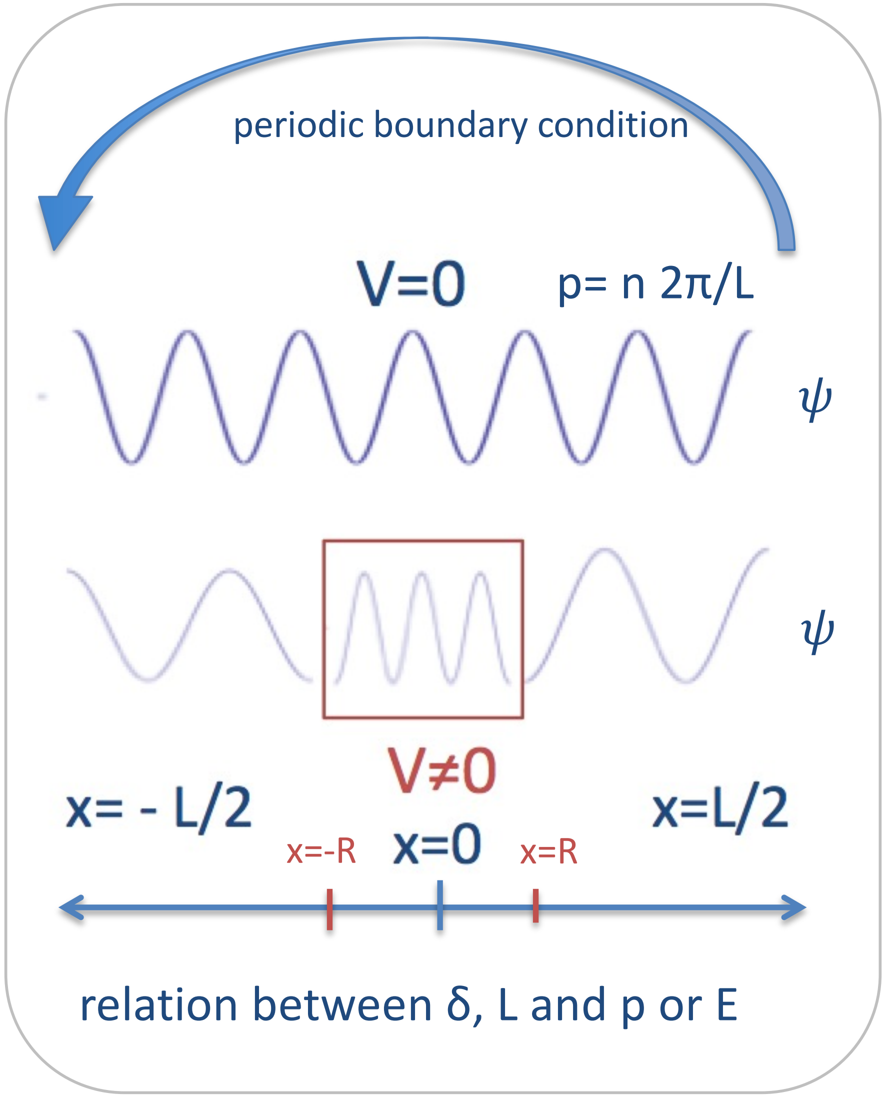
$$\psi(x) = A\cos(p|x| + \delta) = \begin{cases} A\cos(px+\delta) & x > R \\ A\cos(-px+\delta) & x < -R \end{cases}$$

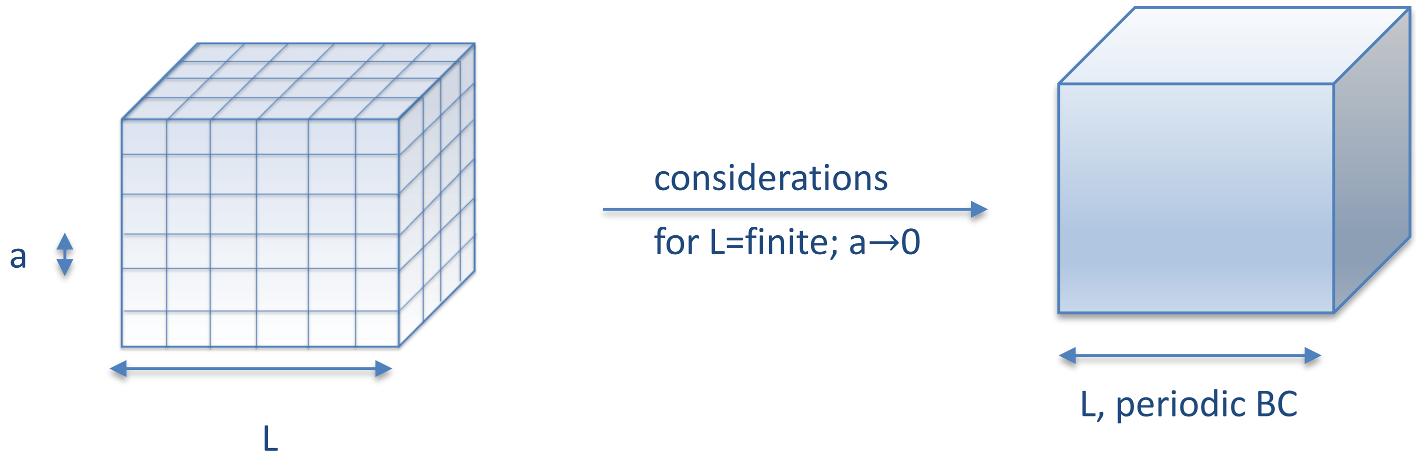
- this form already ensures
  $\psi(L/2) = \psi(-L/2)$
- the other BC:
  $\psi'(L/2) = \psi'(-L/2)$
  this requires
  $A p \sin\left(p\left(\frac{L}{2}\right)+\delta\right) = -Ap\sin\left(-p\left(-\frac{L}{2}\right)+\delta\right)$
  $\rightarrow \psi'(L/2) = 0$, $\sin\left(p\frac{L}{2}+\delta\right) = 0$

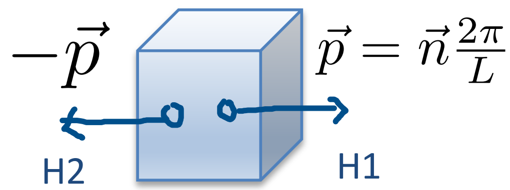
$p\frac{L}{2} + \delta = n\pi$ $\quad \boxed{p = n\frac{2\pi}{L} - \frac{2}{L}\delta}$

relation between δ, L

E=p^2/2m

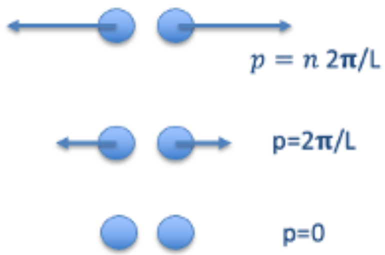




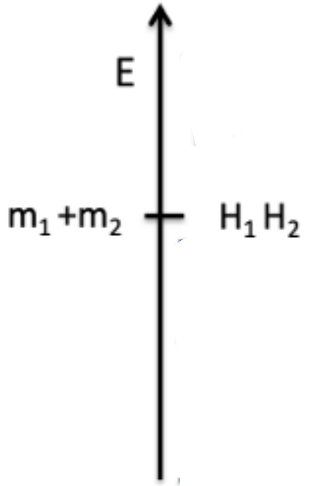


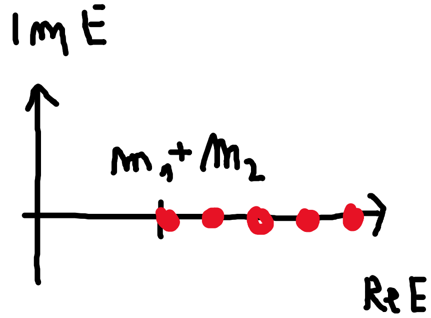
E_n for non-interacting two-hadron system (P=0)

$$E^{n.i.} = \sqrt{m_1^2+\vec{p}^2}+\sqrt{m_2^2+(-\vec{p})^2}$$



$E=m_1+m_2$





Relation between E_n and scattering amlitude in QFT

original: Luscher 1991

Nucl. Phys. B354 (1991) 531-578

mostly following reference:

Kim, Sachrajda, Sharpe 2005

hep-lat/0507006

Differences with respect to Kim, Sachrajda, Sharpe 2005 [KSS]:

(*) KSS considers two identical scalar particles with mass m

I'll consider non-identical degenerate scalar particles with mass m1=m2=m

(*) I'll consider total three-momentum zero, KSS considers general total three momentum

(*) I denote on-shell momemtum p which satisfies $E=2(m^2+p^2)^{1/2}$ [it is cmf momenum as P=0],

KSS denotes on-shell cmf momentum q*

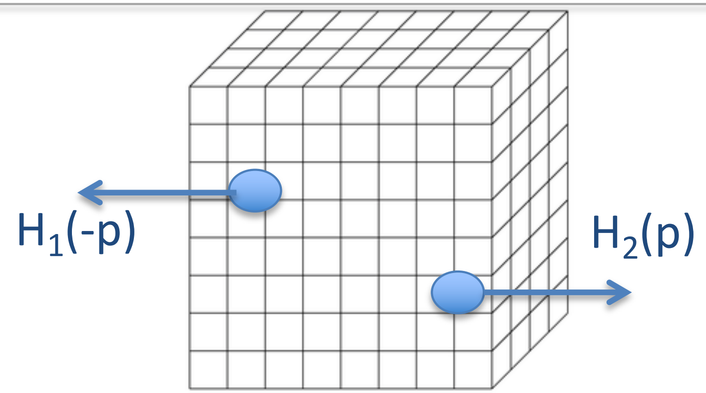
inspired also by

- INT 2021 lectures by Raul Briceno

  https://www.youtube.com/playlist?list=PLDi14w7i5C3Bm3U1IQ4n596UZQhOpr1Cx
- R. Briceno, J. Dudek, R. Young
  1706.06223, Rev. Mod. Phys

Luscher 1991

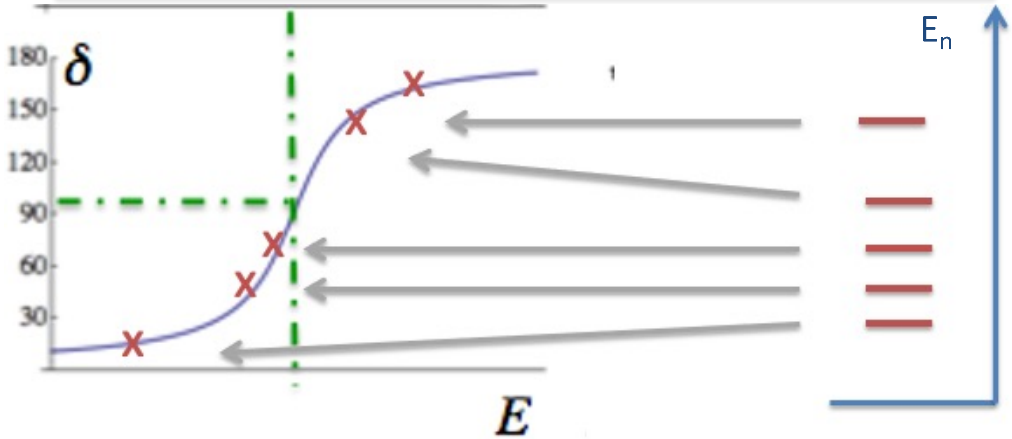
Summarizing: relation between E_n and M∝ T



E=eigen-energy from a lattice simulation

p= on-shell momentum; momentum in region outside V (QM)

$$E = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}$$



Luscher's quantization condition

$$\boxed{\det[\mathcal{M}^{-1}(E) + iF(E)] = 0}$$

$$\mathcal{M}_{l'm',lm}(E) = \mathcal{M}_l(E)\ \delta_{ll'}\delta_{mm'}$$

$$M_l(E) \equiv 8\pi E\ T_l(E)$$

$F_{l'm',lm}(E,L)$ known kinematical fun.

det: in l,m indices

if only one partial-wave l contributes:

$$\mathcal{M}_l^{-1}(E) = -iF_{ll}(E)$$

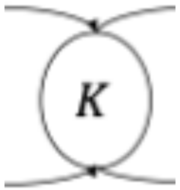
Scatting amplitude in QFT

example: QFT with two scalar non-identical fields with m1=m2=m, CMF: P=(E,0)   E < 3m

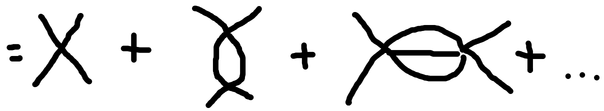
fully dressed and renormalized scalar particle propagator = $\frac{i}{p^2 - m^2 + i\epsilon} + \ldots$

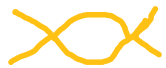
nonsingular terms at $p^2=m^2$

Z=1



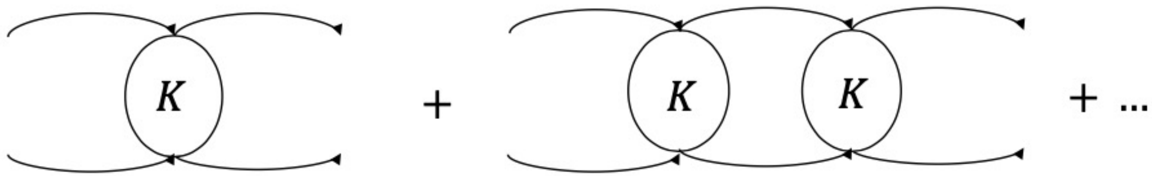
kernel K = sum of amputated two-particle irreducible (2PI) s-channel diagrams





this is not 2PI in s-channel

scattering amplitude $M=$



integral over mom.

E_n and $M(E_n)$ are related via correlation function C in finite volume

derivation in Minkovski space as in Kim, Sachrajda, Sharpe

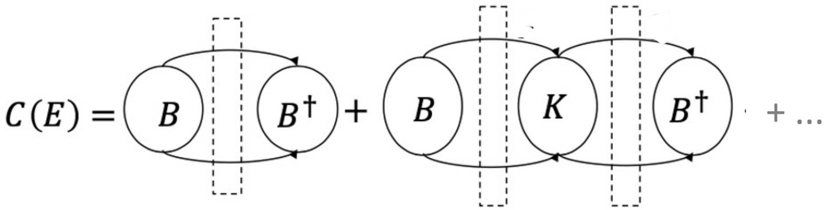
- finite volume C(E) has a pole at E_n : $C(E_n) = \infty$

$$C(t_M) = \sum_n A_n \, e^{-iE_n t_M}$$

$$C(E) \propto \int C(t_M)\, e^{iEt_M}\, dt_M = \sum_n A_n \int e^{-iE_n t_M} e^{iEt_M}\, dt_M$$

$$C(E) \propto \sum_n A_n \, \delta(E - E_n)$$

- finite volume C(E) depends on K, which is related to infinite volume $M \propto T$



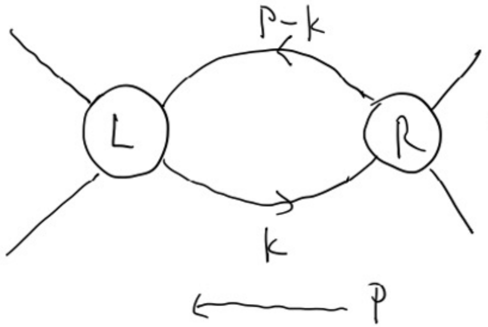
sum over mom.
in the box

this will give us relation between E_n and $M(E_n)$

Bubble diagram: difference between inf. and finite volume

L,R correspond to K or source/sink

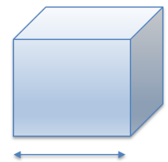
take CMF : $\vec{P} = (E, \vec{0})$ and non-identical particles with $m_1 = m_2$



$$B(\infty) = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi}\, i L(\vec{k})\, \frac{i}{k^2 - m^2 + i\varepsilon}\, \frac{i}{(P-k)^2 - m^2 + i\varepsilon}\, i R(\vec{k})$$

$$B(V) = \frac{1}{L^3} \sum_{\vec{k}} \qquad -\|-$$

$$B(V) - B(\infty) = \left(\frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) \qquad -\|-$$



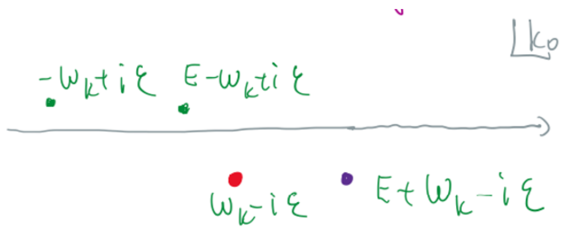
Poisson summation formula

$$\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{i L \vec{l} \cdot \vec{k}} g(\vec{k})$$

for g(k) that has no singularities for real k

$$\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k})$$ up to neglected e^{-mL} corrections

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, i L(\vec{k})\, \frac{i}{k^2 - m^2 + i\varepsilon}\, \frac{i}{(P-k)^2 - m^2 + i\varepsilon}\, i R(\vec{k})$$



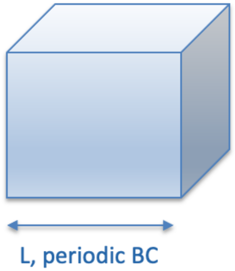
$$\omega_{\vec{k}} \equiv \sqrt{m^2 + \vec{k}^2}$$

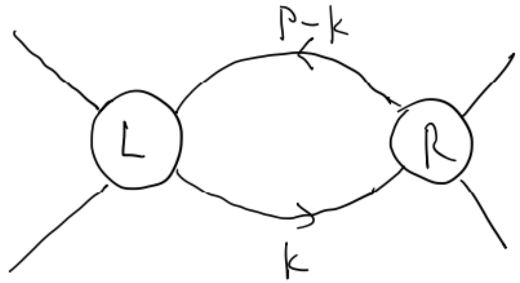
closing contour downwards:

blue pole: no singularities in physical region of interest:

this can lead only to exp. small corrections which we neglect

Bubble diagram, cont'

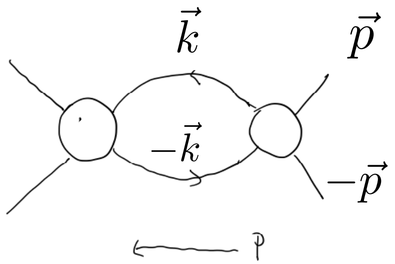




P=(E,0)

$$B(V) - B(\infty) = \left(\frac{1}{L^3}\sum_{\vec k} - \int\frac{d\vec k}{(2\pi)^3}\right) \frac{1}{2\omega_k}\, iL(k)\, \frac{i}{(E-k^0)^2 - \vec k^2 - m^2 + i\epsilon}\, iR(k)\Big|_{k^0 = i\omega_k} + O(e^{-mL})$$

residuum at •   residuum at •



$$\frac{1}{(E-\omega_k)^2-\omega_k^2+i\epsilon} = \frac{1}{E-\omega_k+\omega_k}\,\frac{1}{E-\omega_k-\omega_k+i\epsilon}$$

$$a^2-b^2 = (a+b)(a-b)$$

only pole is at $E = 2\omega_{\vec k}$

when both particles come on-shell.

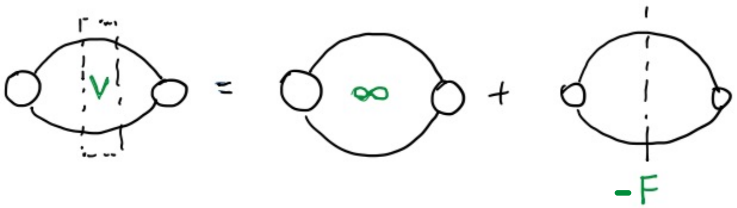
This happens in physical scattering

for k that equals on-shell mom $|\vec k| = |\vec p|$

$E = \sqrt{m^2+\vec p^2} + \sqrt{m^2+\vec p^2}$ $p \equiv |\vec p|$

definition of on-shell momentum p

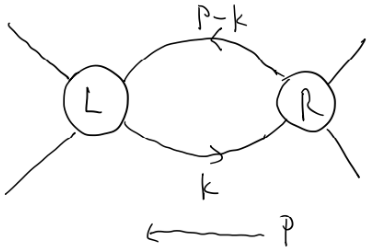
finite sum_k.   int   finite V correction



diagrams where an intermediate two-particle state can go on-shell play the dominant role in determining the dependence of the correlation function on the finite-volume. Qualitatively this can be understood by recognizing that on-shell particles can propagate over arbitrary distances and hence sample the boundaries of the volume, and the quantitative manifestation of this will be pole singularities at energies corresponding to allowed free two-particle states. Diagrams in which the intermediate two-particle state cannot go on shell can be shown to contribute at a level which is exponentially suppressed $\sim e^{-m_\pi L}$, and these can be neglected for volumes $L \gg m_\pi^{-1}$.

R. Briceno, J. Dudek, R. Young, 1706.06223

Bubble diagram, cont'



P=(E,0)

$$B(\infty)=\int\frac{d\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_k}\,iL(k)\,\frac{i}{(E-\omega_k)^2-\vec{k}^2-m^2+i\varepsilon}\,iR(k)$$

$$\frac{1}{(E-\omega_k)^2-\omega_k^2+i\varepsilon}=\frac{1}{E-\omega_k+\omega_k}\;\frac{1}{E-\omega_k-\omega_k+i\varepsilon}$$

$$a^2-b^2=(a+b)(a-b)$$

$$B(\infty)=\sum_{\substack{lm\\ l'm'}} i\,\mathcal{L}_{l'm'}(p)\left\{\int\frac{d\vec{k}}{(2\pi)^3}\,\frac{i\,4\pi\, Y_{l'm'}(\hat{k})\,Y_{lm}(\hat{k})}{2\omega_k\, E\,(E-2\omega_k+i\varepsilon)}\right\} i\,\mathcal{R}_{lm}(p)$$

$$L(\vec{k})=\sum_{l'm'}\sqrt{4\pi}\,Y_{l'm'}(\hat{k})\,\mathcal{L}_{l'm'}(|\vec{k}|)$$

$$R(\vec{k})=\sum_{lm}\sqrt{4\pi}\,Y_{lm}(\hat{k})\,\mathcal{R}_{lm}(|\vec{k}|)$$

pole in denominator enforces k=p

$$\frac{1}{E-2\omega_k+i\varepsilon}=\mathrm{P.V.}\frac{1}{E-2\omega_k}-i\pi\,\delta(E-2\omega_k)$$

smooth

on-shell

renders Im(B)

$$\mathrm{Im}(B(\infty))\propto\int\frac{d\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_k\,2\omega_k}\,\delta(E-2\omega_k)\propto \text{2-particle phase space}\propto\frac{p}{E}$$

$$B(L)-B(\infty)=\sum_{\substack{lm\\ l'm'}} i\,\mathcal{L}_{l'm'}(p)\underbrace{\left\{\left(\frac{1}{L^3}\sum_{\vec{k}}-\int\frac{d^3\vec{k}}{(2\pi)^3}\right)\frac{i\,4\pi\, Y_{l'm'}(\hat{k})\,Y_{lm}(\hat{k})}{2\omega_k\, E\,(E-2\omega_k+i\varepsilon)}\right\}}_{i^2 F_{l'm',lm}(E,L)\,=\,-F} i\,\mathcal{R}_{lm}(p)$$

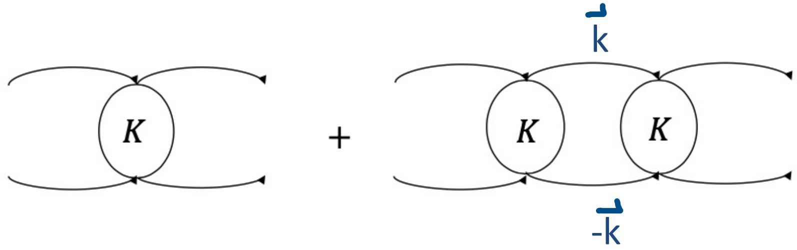
F = known kinematical matrix function

same as in Kim, Sachrajda, Sharpe 2005 (eqs 48,49)

omitting i from each of two props in definition of F

Scattering amplitude for V=inf

M=



+ ...

E = $\sqrt{m^2+p^2}+\sqrt{m^2+p^2}$

$$M_l(E) \equiv 8\pi E\; T_l(E)$$

$$iM = iK + iK\underbrace{(\mathrm{real} + i\tfrac{p}{8\pi E})}_{\text{bubble}}iK + \ldots = \frac{i}{\mathrm{real} - i\frac{p}{8\pi E}}\;, \qquad T = \frac{1}{\mathrm{real} - ip}$$

$$= \frac{1}{\underbrace{p\cot\delta}_{\text{real}} - ip}$$

for real E above and below threshold

Homework 8:

collect real pieces and

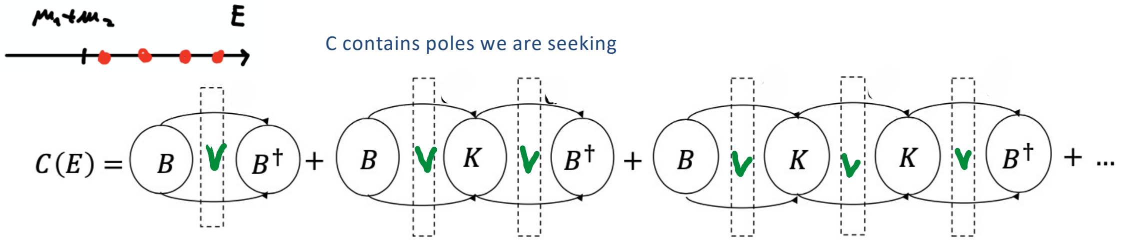
perform geometric series

The imaginary part of M^{-1} is uniquely rendered from the kinematical region when both particles are on-shell :

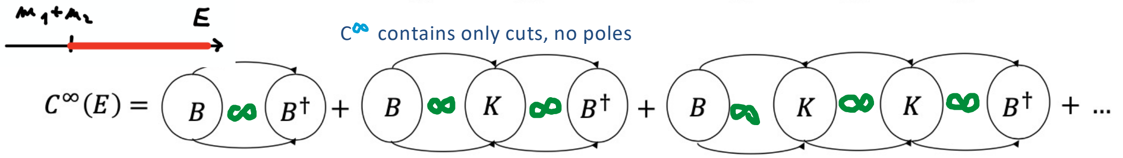
Determining poles of C(E)

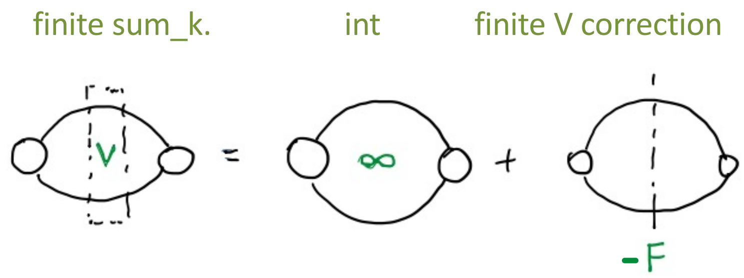
$C^{FV}(E_n) = \infty$

C contains poles we are seeking



C^∞ contains only cuts, no poles

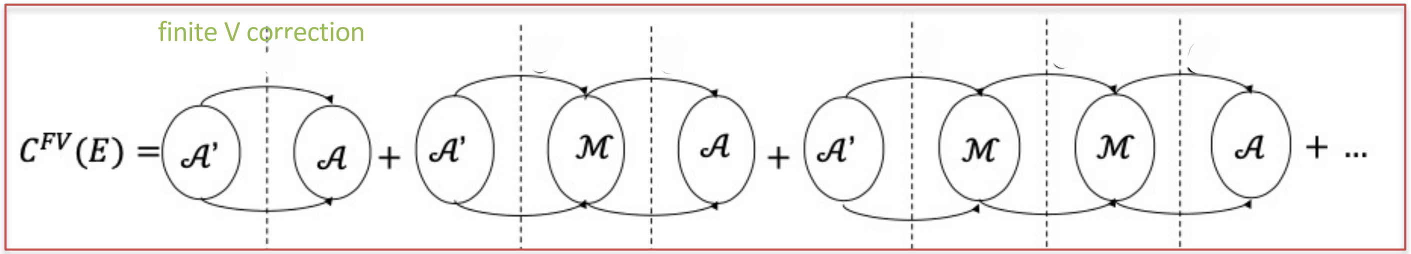




$$C^{FV}(E) = C(E) - C^\infty(E)$$

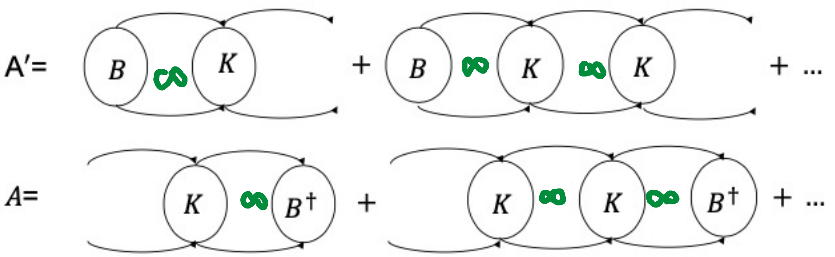
C^{FV} contains same poles as C

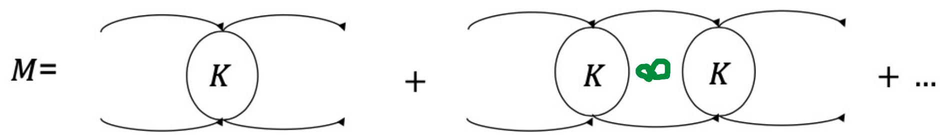
By inserting the above sum to C,
and using definitions of A, A' and M:
one can (pictorially) show



Homework :

try to understand diagramatically



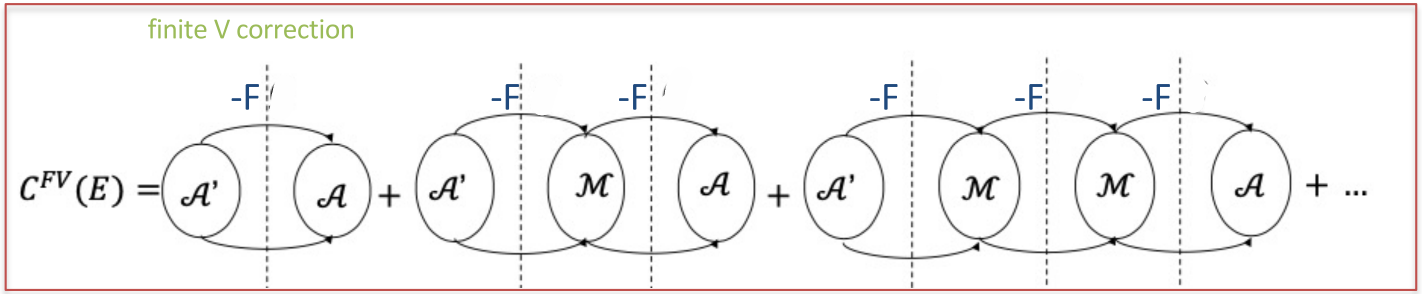


K = sum of all amputated two-particle
irreducible scattering diagrams in s−channel

M = scattering amplitude we are after

Quantization condition for poles of $C^{FV}(E)$

$C^{FV}(E_n) = \infty$



geometric sum

$$C^{FV}(L,E) \propto A' \left[-F + F i\mathcal{M} F + \ldots\right] A = A'\, F \sum_{n=0}^{\infty} [-i\mathcal{M}F]^n\, A = A' F \frac{1}{1+i\mathcal{M}F}\, A = A' F \mathcal{M} \frac{1}{\mathcal{M}^{-1}+iF}\, A$$

$$\frac{1}{\mathcal{M}^{-1}+iF} \propto \frac{1}{\det[\mathcal{M}^{-1}+iF]}$$

pole of C(E) at E=E_n where

$$\boxed{\det[\mathcal{M}^{-1}(E) + iF(E)] = 0}$$

quantization condition

= Luscher's relation

$\mathcal{M}$ scattering amplitude $\quad \mathcal{M}_{l'm',lm}(E) = \mathcal{M}_l(E)\ \delta_{ll'}\delta_{mm'} \quad M_l(E) \equiv 8\pi E\ T_l(E)$

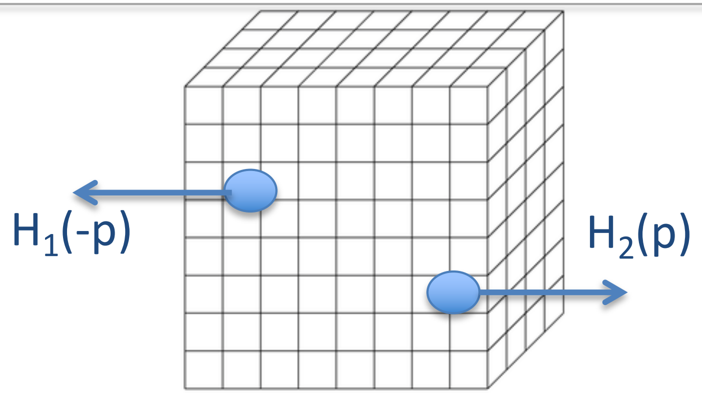
F known finite volume correction to the loop function $F_{l'm',lm}(E,L)$

both are matrices in space of partial waves: M diagonal for spin-less particles (since J=l is a good quantum number)

F nondiagonal in general

Luscher 1991

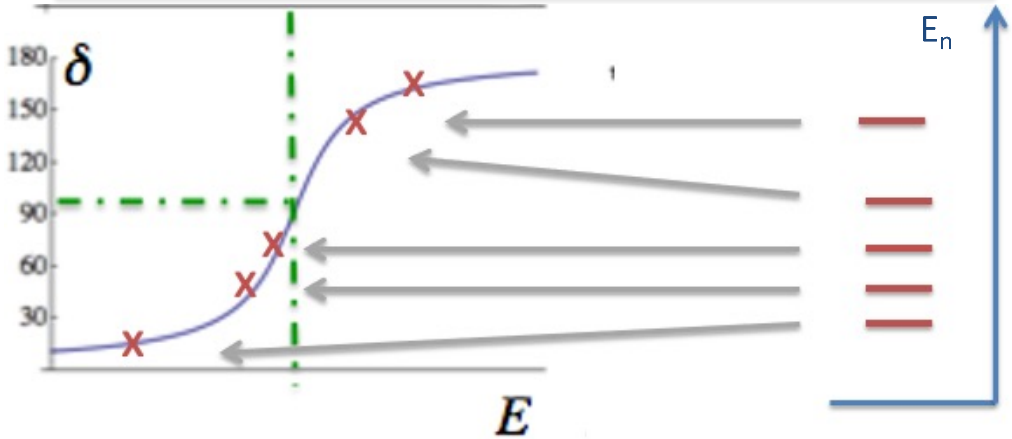
Summarizing: relation between E_n and M$\propto T$



E=eigen-energy from a lattice simulation

p= on-shell momentum; momentum in region outside V (QM)

$$E = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}$$



Luscher's quantization condition

$$\boxed{\det[\mathcal{M}^{-1}(E) + iF(E)] = 0}$$

$$\mathcal{M}_{l'm',lm}(E) = \mathcal{M}_l(E)\ \delta_{ll'}\delta_{mm'}$$

$$M_l(E) \equiv 8\pi E\ T_l(E)$$

$F_{l'm',lm}(E,L)$ known kinematical fun.

(TwoHadronsInBox package, Morningstar et al)

det: in l,m indices

many generalizations; most general (particles with arbitrary spin, total momentum, several two-particle channels):

Briceno, PRD 89, 074507 (2014)

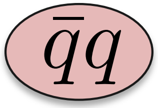
if only one partial-wave l contributes:

$$\mathcal{M}_l^{-1}(E) = -iF_{ll}(E)$$

only l=0: $p \cdot \cot\delta_0(p) = \dfrac{Z_{00}(1;(\frac{pL}{2\pi})^2)}{\sqrt{\pi}L}$

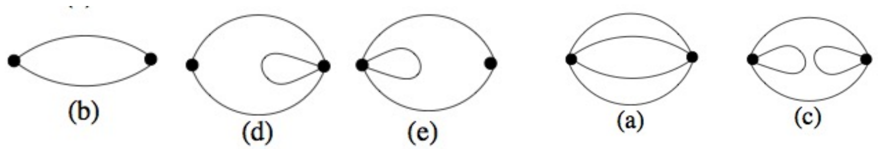
$\pi\,\pi$ scattering in ρ-channel, $J^{PC}=1^{--}$, I=1 , l=1

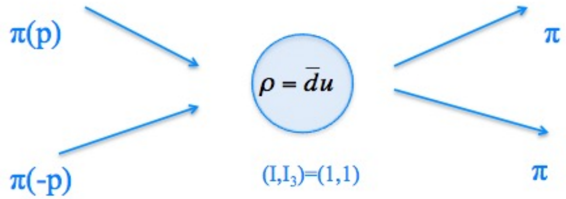
Lang, Mohler, S.P., Vidmar
PRD 2011, m_π=266 MeV, Nf=2
single volume N_L=16



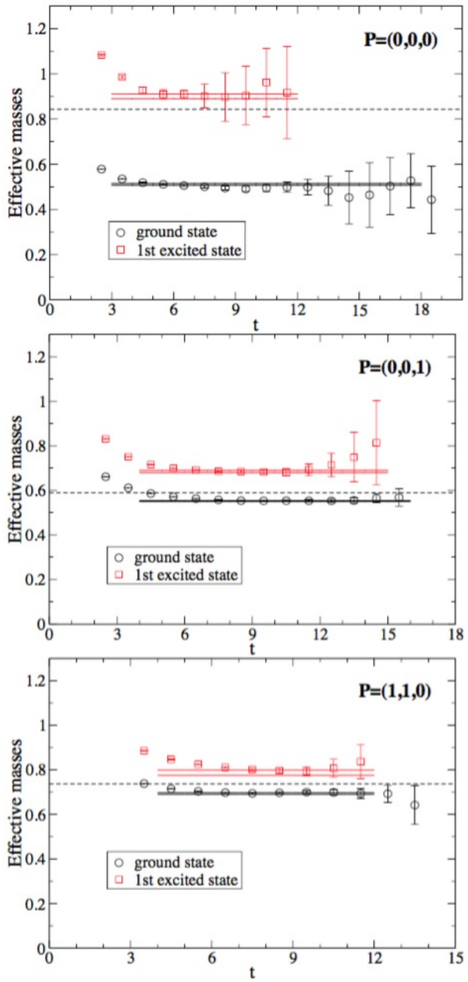
note: this "simple" calculation
is chosen for pedagogical purpose,
now calculations are more advanced

$$C_{jk}(t)=\langle 0|O_j(t)\,O_k^+(0)|0\rangle,\quad O=\bar{q}q,\ (\bar{q}q)(\bar{q}q)=\pi\pi,\quad 1^{--}$$





eigen-energies for NL=16



$\pi\pi$

$\pi\pi$

$\pi\pi$

dashed
lines: E^{ni}

interpolators

$\bar{q}\ \Gamma\ q\ (000)$ → always several of those

$\pi(-001)\pi(001)$

↓ mom. proj.

$\bar{q}\ \Gamma\ q\ (001)$

$\pi(000)\pi(001)$

$\bar{q}\ \Gamma\ q\ (110)$

$\pi(000)\pi(110)$

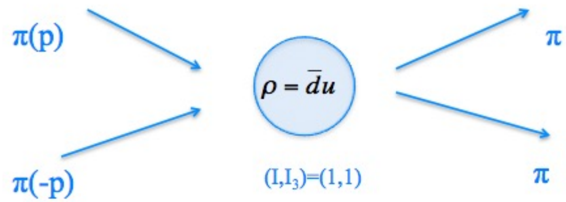
note: En have to be determined so accurately
that energy shifts can be extracted reliably

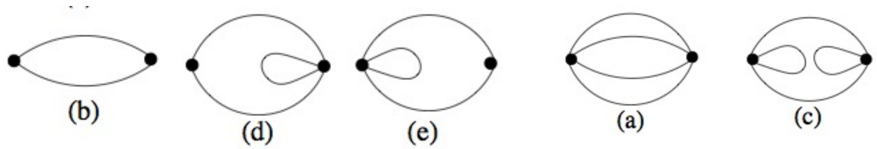
first study lattice study which considered scattering
using distillaiton method to calculate Wick contractions

$\pi\,\pi$ scattering in ρ-channel, $J^{PC}=1^{--}$, I=1 , l=1

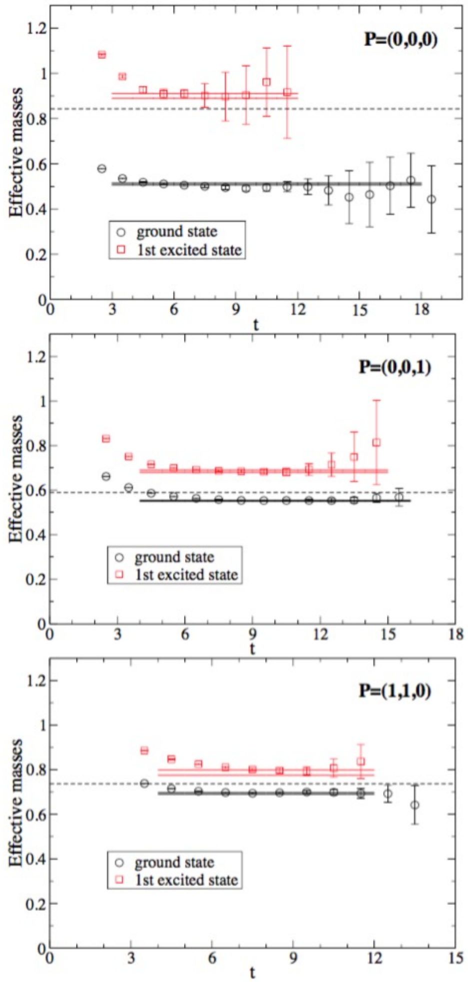
Lang, Mohler, S.P., Vidmar
PRD 2011, m_π=266 MeV, Nf=2
single volume N_L=16

$$C_{jk}(t)=\langle 0|O_j(t)\,O_k^{+}(0)|0\rangle,\quad O=\bar{q}q,\ (\bar{q}q)(\bar{q}q)=\pi\pi,\quad 1^{--}$$





eigen-energies for NL=16



$$\mathcal{M}_l^{-1}(E_{cm})=-iF_{ll}(E_{cm})$$

l=1 ⟹

Luscher's relation

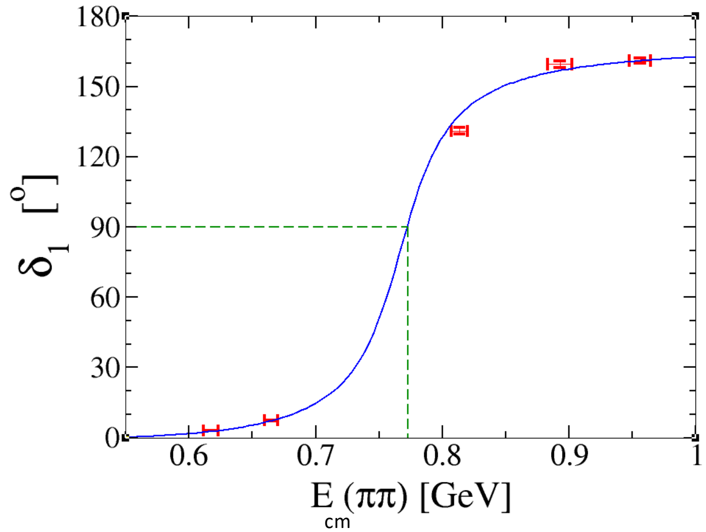
for P=0, P ≠ 0

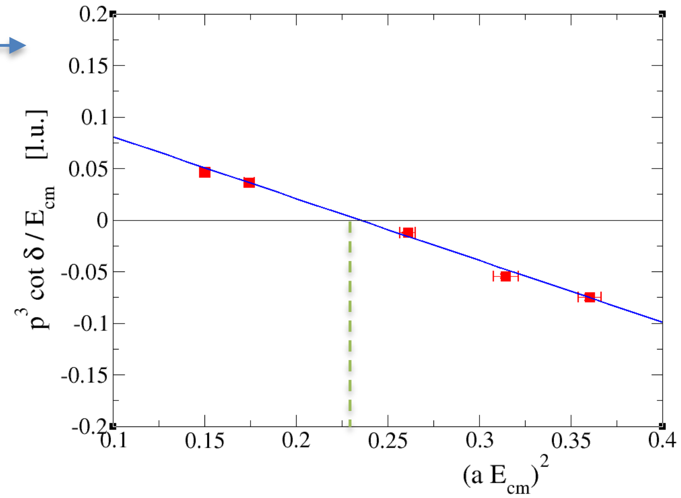
$$\mathcal{M}=\frac{1}{8\pi E(p\cot\delta-ip)}$$

$$\frac{p^{2l+1}\cot\delta}{E_{cm}}=\frac{m_R^2-E_{cm}^2}{g^2}$$

$$\Gamma(E_{cm})=g^2\,\frac{p^{2l+1}}{E_{cm}^2}$$

ρ meson	Mass [MeV]	$g_{\rho\pi\pi}\equiv\sqrt{6\pi}\,g$
Lat	772 ± 6 ± 8	5.61 ± 0.12
Exp.	775	5.97





E_{cm}=772 MeV

Scalar charmed meson

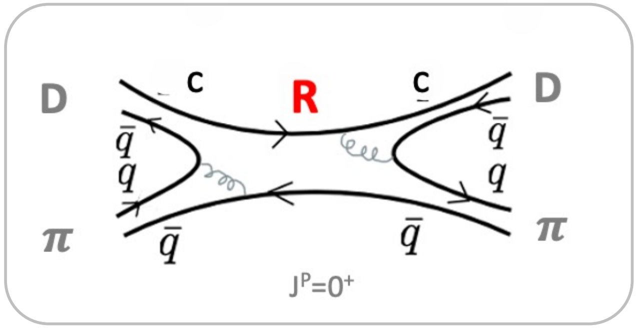
$c\bar{d}$, $c\bar{d}q\bar{q}$ $J^P = 0^+$

not explicity exotic;

it's low mass indicates non-conventional states in this sector

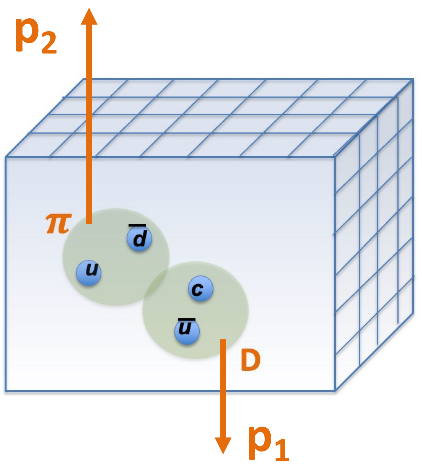
HadSpec, Gayer et al, 2102.04973

$m_\pi \approx 240$ MeV

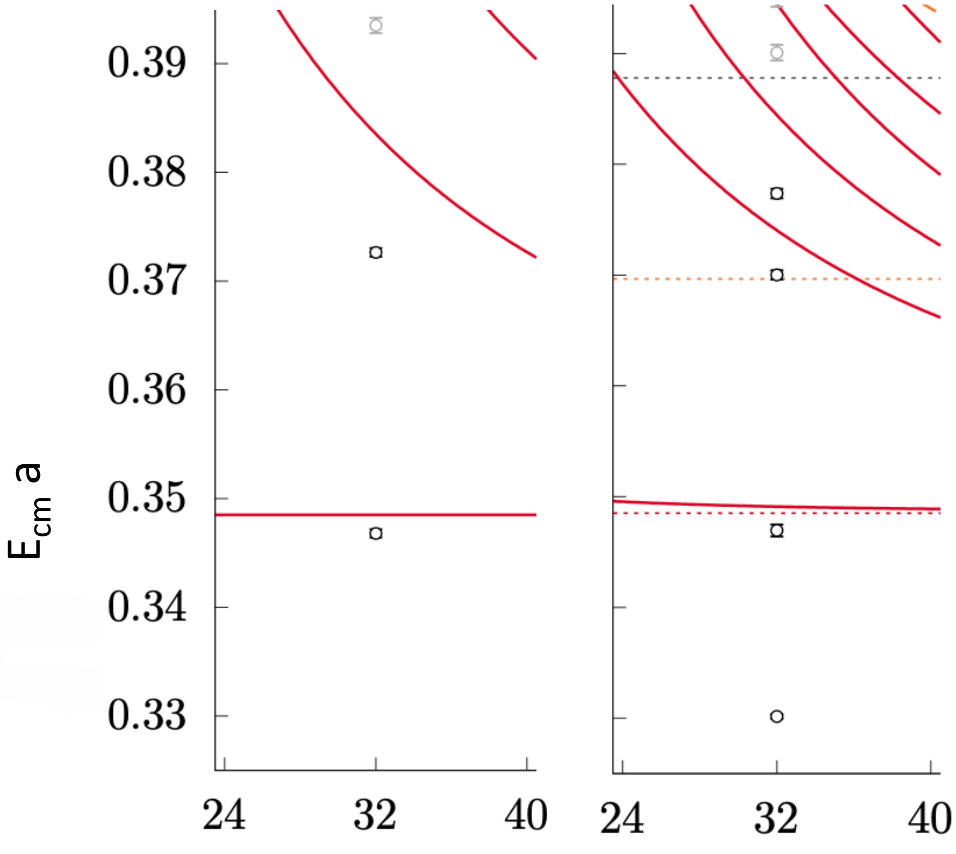


$$O \sim (\bar{u}\Gamma_1 c)_{\vec{p}_1} (\bar{d}\Gamma_2 u)_{\vec{p}_2} + ..$$

$$\sim D(\vec{p}_1) \quad \pi(\vec{p}_2)$$



red lines: $E^{\text{non-int.}} = \sqrt{m_D^2 + p_1{}^2} + \sqrt{m_\pi^2 + p_2{}^2}$ $\vec{p}_i = \vec{n}_i \frac{2\pi}{L}$



(a) $[000]A_1^+$ **(a)** $[100]A_1$

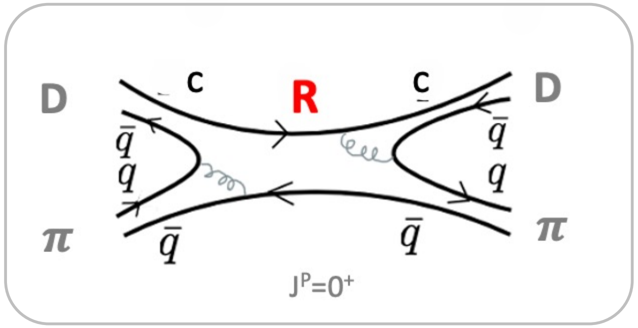
+ other total momenta and irreps

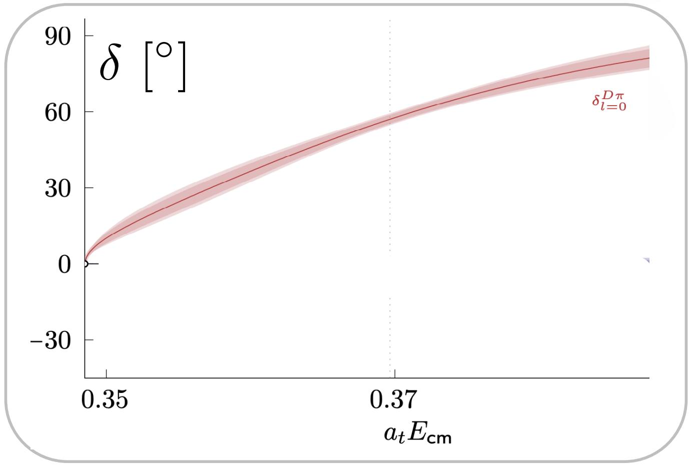
Scalar charmed meson, cont'

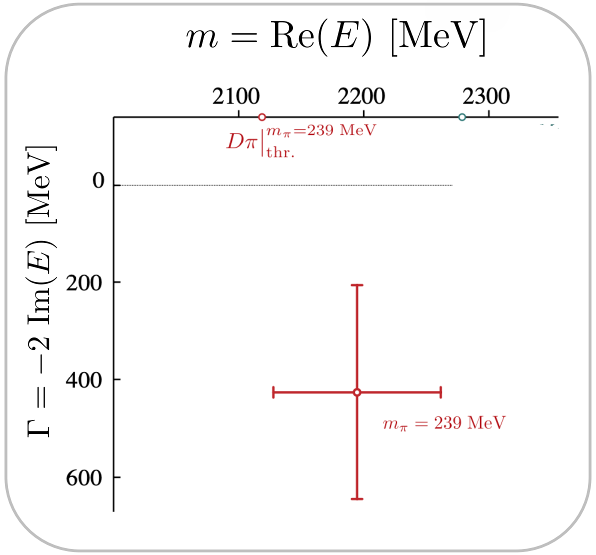
$c\bar{d}$, $c\bar{d}q\bar{q}$

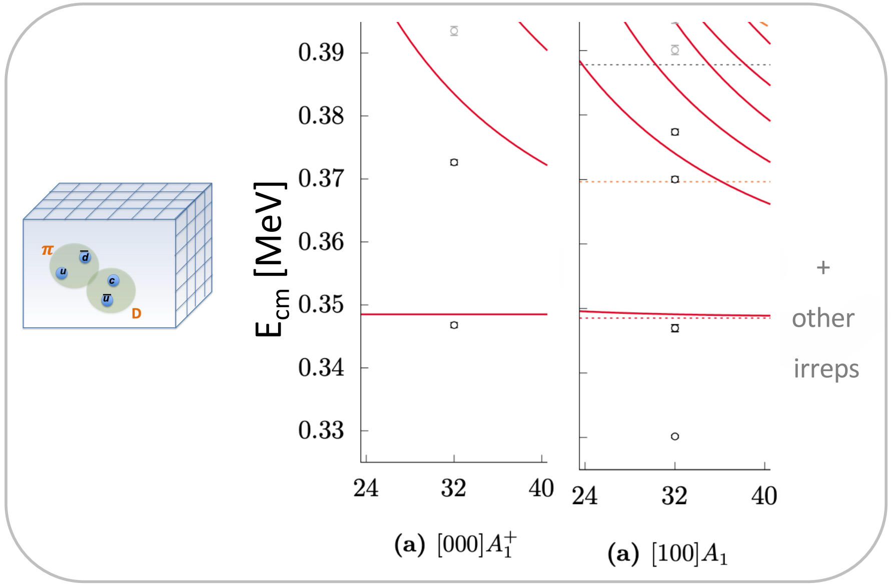
HadSpec, Gayer et al, 2102.04973

$m_\pi \approx 240$ MeV



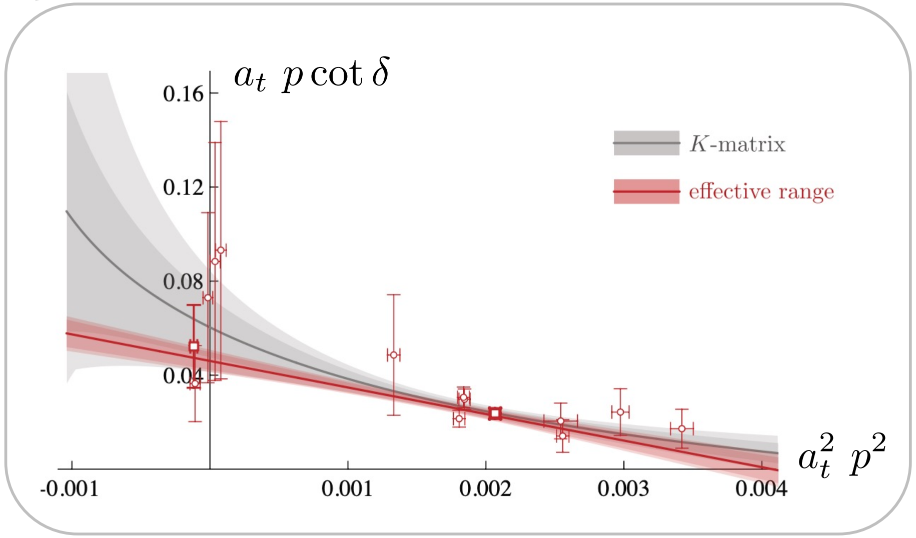






Luscher's relation

$$\frac{2}{E}\, p \cot\delta(E) = -F(E,\vec{P},L)$$



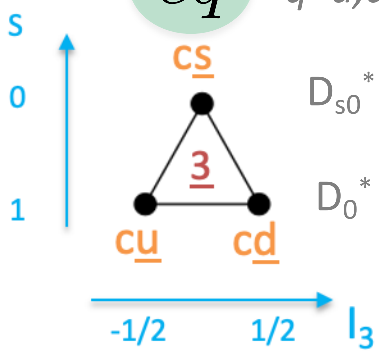
Scalar heavy-light mesons $J^P = 0^+$

Scattering on the lattice
~~SU(3)~~$_F$
S=1 Mohler et al, 1308.3175, PRL
Lang et al, 1403.8103, PRD
RQCD, 1706.01247, PRD
HadSpec 2008.06432, JHEP
S=0 Mohler et al. 1208.4059, PRD
HadSpec, 1607.07093, JHEP
HadSpec 2102.04973, JHEP
S=-1 HadSpec, 2008.06432, JHEP

SU(3)$_F$: Gregory et al, 2106.15391
attraction in 6, repulsion in 15

Conventional quark model

$c\bar{q}$ q=u,d,s



new paradigm supported by:

- lattice
- ChPT+HQET, UChPT
- reanalysis of exp data
- states circled by blue feature in the spectrum

New paradigm

Lutz et al, 2003 PLB, 2209.10601
Du et al, 1712.07957, PRD
Albaladejo et al, 1610.06727

$c\bar{q}$ + $c\bar{q}\ q\bar{q}$ q=u,d,s n=u,d

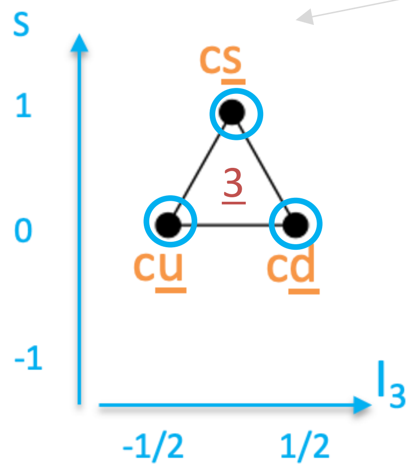
$\underline{3} \otimes 8 = \underline{3} \oplus 6 \oplus 15$ SU(3)$_F$

most attractive / attractive / repulsive

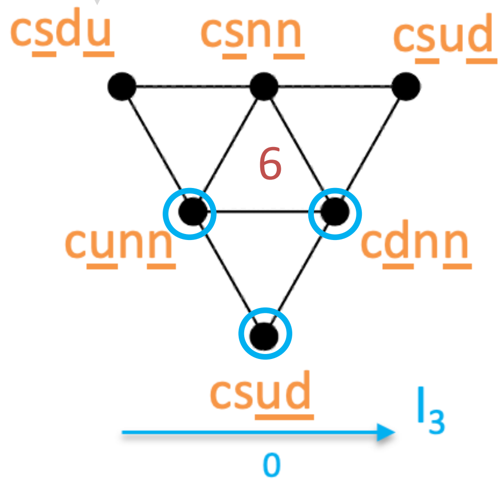
$D_{s0}(2317)$: 70-100% DK molecule 2.3 GeV

lat: 2.1-2.2 GeV (pole)

PDG: 2.3 GeV (BW)



mix
~~SU(3)~~$_F$



mixes with 15

2.4-2.5 GeV r

eanalysis of lat 1607.07093 by Albaladejo 1610.06727

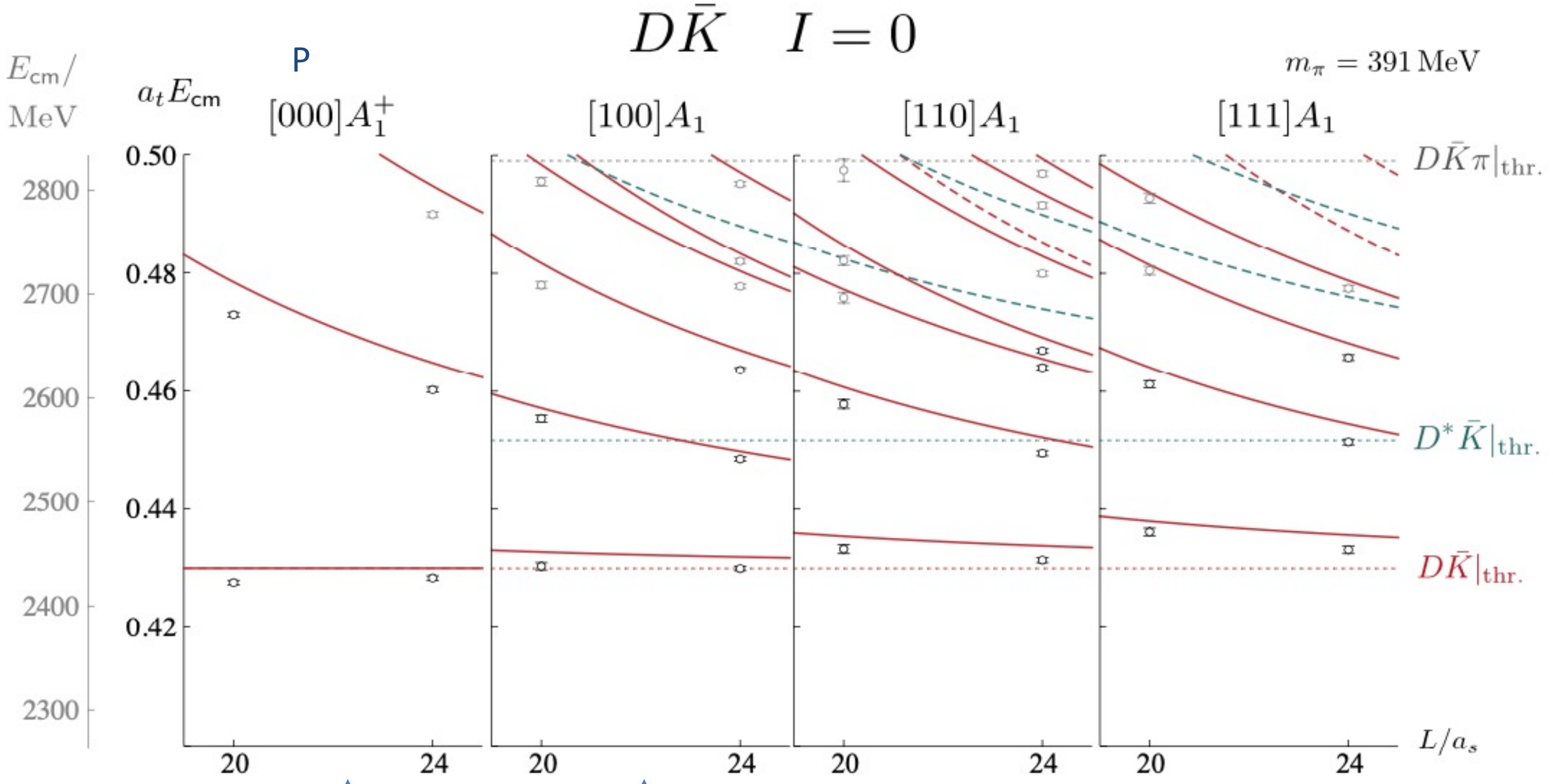
virtual bound state
HadSpec 2008.06432
partner of X(2900) [LHCb] ?

Channel with exotic flavor D $\underline{K}$

$c\bar{u}s\bar{d}$

$J^P=0^+$, $I=0$

HadSpec. Coll, 2008.06432



+ 9 more P/irreps

operators

D(0,0,0)$\underline{K}$(0,0,0)
D(0,0,1)$\underline{K}$(0,0,-1)+..
D(0,1,1)$\underline{K}$(0,-1,-1)+..
...

D(1,0,0)$\underline{K}$(0,0,0)
D(0,0,0)$\underline{K}$(1,0,0)+..
D(1,1,0)$\underline{K}$(0,-1,0)+..
D(1,1,1)$\underline{K}$(0,-1,-1)+..
...

certain irrep gets contribution fom multiple partial waves (odd and even) for P $\neq$ 0 since mD $\neq$mK

D K

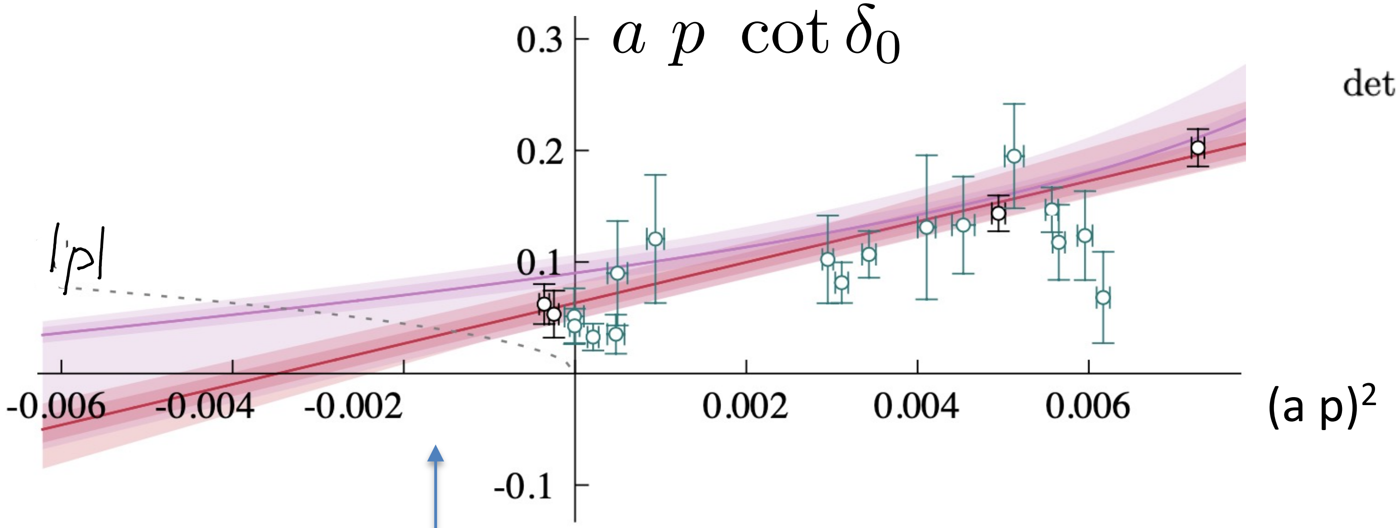
$J^P=0^+$, I=0, l=0

$c\bar{u}s\bar{d}$

HadSpec. Coll, 2008.06432

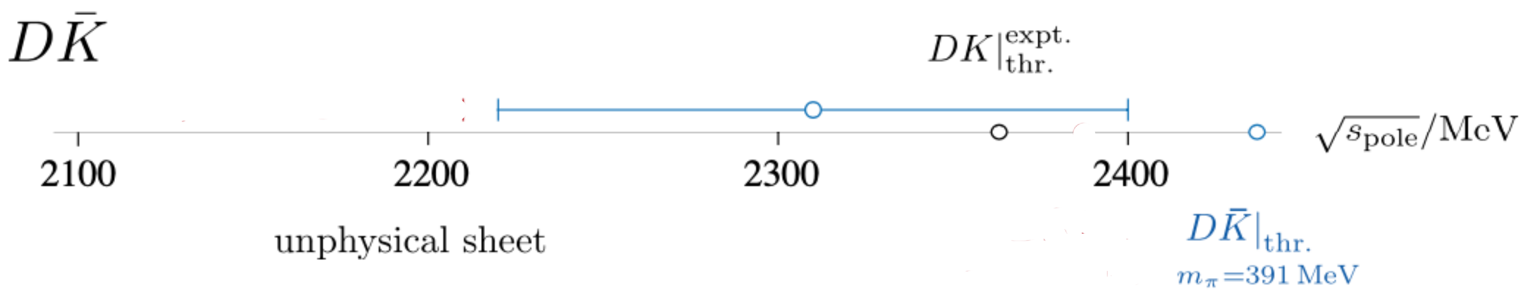
$$\mathcal{M} = \frac{1}{8\pi E(p\cot\delta - ip)}$$

$$\det[\mathcal{M}^{-1}(E_{cm}) + iF(E_{cm})] = 0$$



Virtual bound state: Riemann sheet II: p= - i |p|

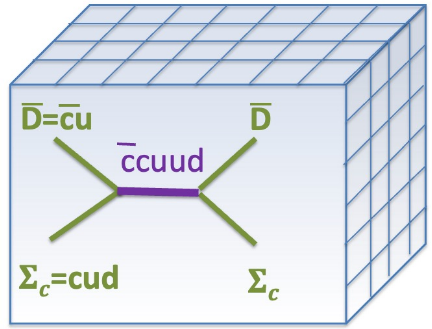
p cot(del) = i p = |p|



P_c

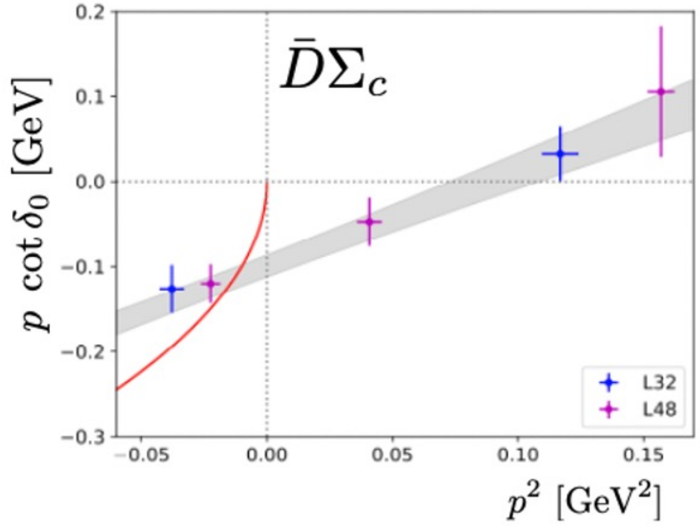
$$\bar{c}cuud \;\rightarrow (\bar{c}u)(cud), \ldots$$
$$\not\rightarrow (\bar{c}c)(uud)$$

caution: coupling to charmonium+proton omitted

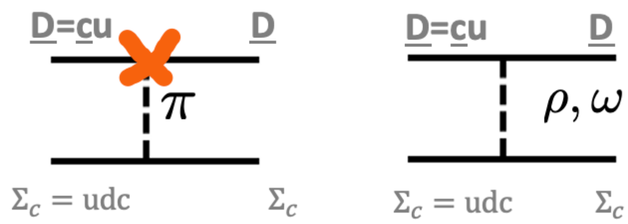


H. Xiang et al., 2210.08555 $m_\pi \approx 294$ MeV

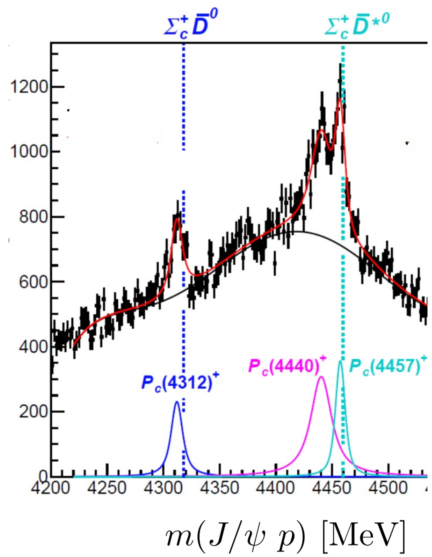
$\underline{D}\Sigma_c$ in s-wave $J^P=1/2^-$



Wu, Molina, Oset, Zou, PRL 2010



LHCb 2019



$$T \propto \frac{1}{p\cot\delta - ip}\ , \quad p\cot\delta = \frac{1}{a_0} + \frac{1}{2}r_0\, p^2$$

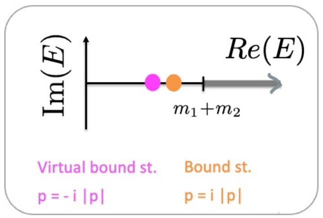
$$\frac{1}{a_0} + \frac{1}{2}r_0 p^2 - ip = 0 \;\rightarrow\; p_b = i|p_b|$$

$$m_{P_c} = \sqrt{m_D^2 + p_b^2} + \sqrt{m_{\Sigma_c}^2 + p_b^2}$$

$$m_{P_c} - (m_D + m_{\Sigma_c}) = -6 \pm 3 \text{ MeV}$$

pole in T:

real bound st.



T_{bc} : next exciting discovery from exp ?

$$bc\bar{u}\bar{d}$$

$$I=0,\ J^P=1^+,0^+$$

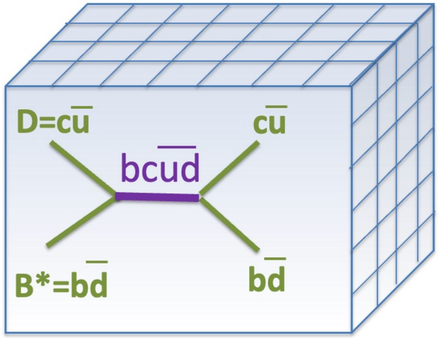
Alexandrou et al, 2312.02925 PRL

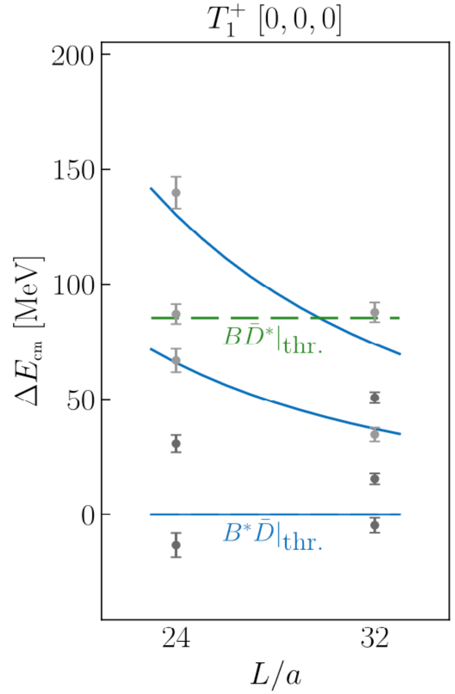
$m_\pi \approx 220 MeV$

$$O \sim (\bar{u}b)(\bar{d}c),\ [bc][\bar{u}\bar{d}]$$

B* D

B D*



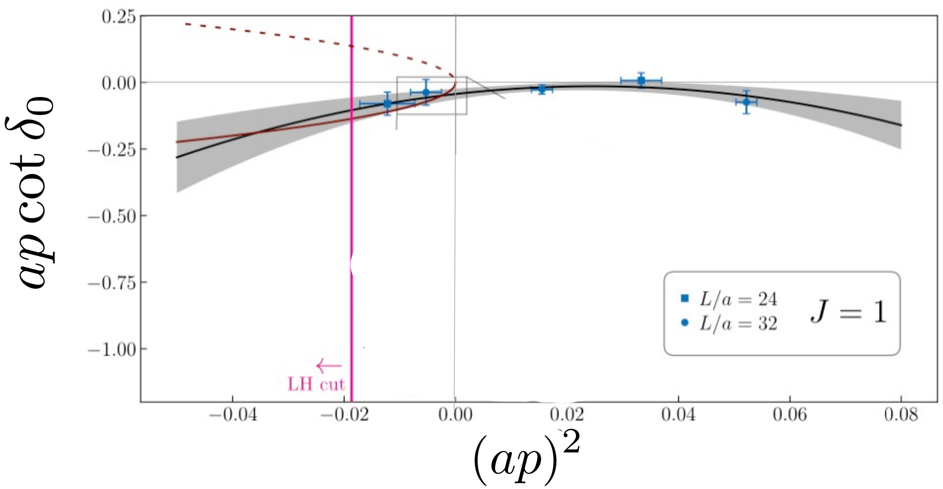


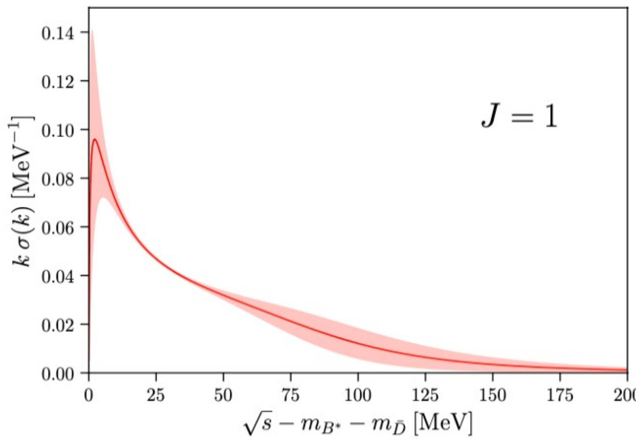
E

Luscher's rel.

δ(E), T(E)

$$T_0 \propto \frac{1}{p\cot\delta_0 - ip}$$





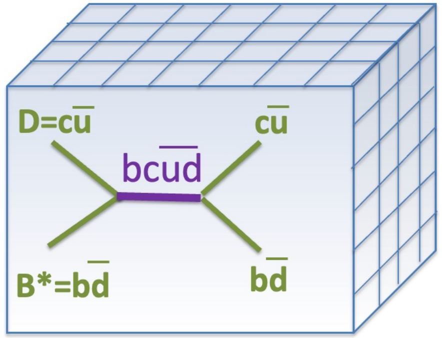
$$m_{T_{bc}} - m_{B^*} - m_D = -2.4^{+2.0}_{-0.7}\ \text{MeV}$$

$$m_R - m_{B^*} - m_D = 67 \pm 24\ \text{MeV} \quad \Gamma_R = 132 \pm 32$$

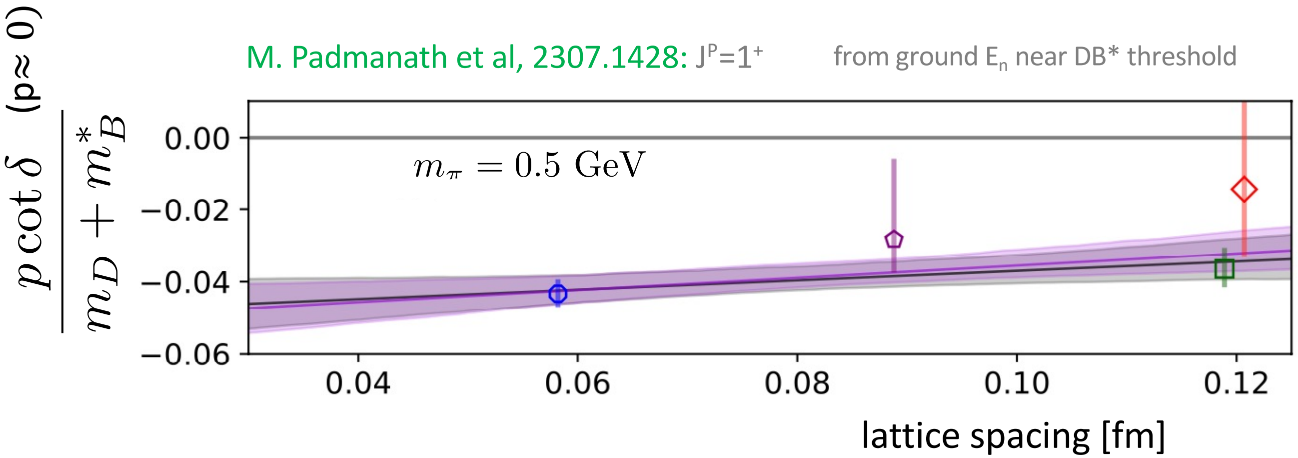
T_{bc} : next exciting discovery from exp ?

$I=0,\ J^P=1^+,0^+$ $\quad bc\bar{u}\bar{d}$

$O \sim (\bar{u}b)(\bar{d}c),\ \ [bc][\bar{u}\bar{d}]$



M. Padmanath et al, 2307.1428: $J^P=1^+$ from ground E_n near DB* threshold

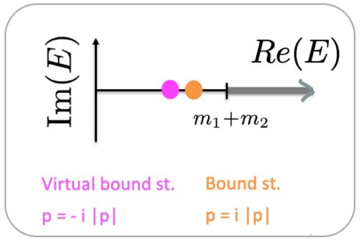


$$T \propto \frac{1}{p\cot\delta - ip}\ ,\quad p\cot\delta \overset{\text{approx}}{=} \frac{1}{a_0}$$ $\quad a_0<0$

$$\frac{1}{a_0} - ip = 0\ \rightarrow\ p_b = -i\frac{1}{a_0} = i\ \left|\frac{1}{a_0}\right|$$

$$m_{T_{bc}} - (m_D + m_{B^*}) = -43(^{+6}_{-7})(^{+14}_{-24})\ \text{MeV}$$

pole in T : real bound st.



after continuum extrap. and chiral extrap. from $m_\pi = 0.5 - 1$ GeV

T_{cc} from LHCb experiment

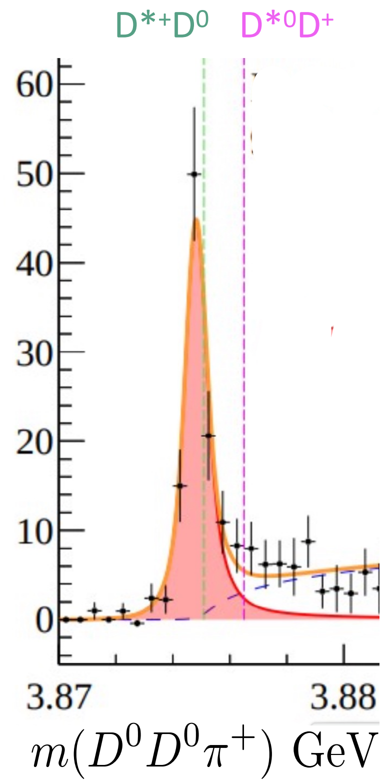
$$D^* \to D\pi$$

$$m_{\pi^0} \simeq 135 \text{ MeV}$$
$$m_{D^{*+}} - m_{D^+} \simeq 140 \text{ MeV}$$

$cc\bar{d}\bar{u}$

I=0, $J^P=1^+$ (most likely)

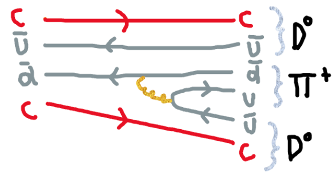
The longest lived exotic hadron ever discovered



$$\delta m = m - (m_{D^{*+}} + m_{D^0})$$

$$\delta m_{pole} = -0.36 \pm 0.04 \text{ MeV}$$

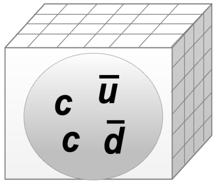
LHCb 2109.01038, 2109.01056, Nature Physics



Omitting $D^* \to D\pi, \; T_{cc} \to DD\pi$

T_{cc} would be a bound state

T_{cc} from lattice



all analyzed in 2402.14715, PRD
Collins, Nefediev, Padmanath , SP;

all simulations :

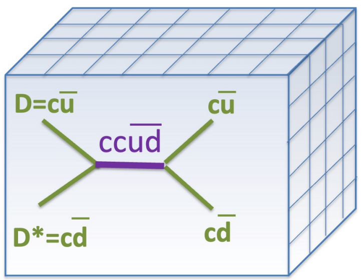
$$m_u = m_d > m_{u,d}^{ph} \qquad D^* \not\to D\pi$$

single lattice spacing

(J. Green et al are exploring several lattice spacings, lat 2023, unpublished)

mc	mpi	L	ensmbles	ref.
five values m_D=1.7–2.4 GeV	280 MeV	~ 2.1, 2.8 fm	CLS Nf=2+1	our, 2402.14715, PRD, Padmanath &SP, 2202.10110 PRL eigenenergies

mc	mpi	L	ensembles	ref.
~ physical	146 MeV	~ 8 fm	Nf=2+1	HALQCD, 2302.04505, PRL HALQCD potentials
~ physical	280 MeV	~ 2.1, 2.8 fm	Nf=2+1, CLS	our, 2402.14715, PRD eigenenergies
~ physical	348 MeV	~ 2.4 fm	Nf=2	CLQCD, 2206.06186, PLB eigenenergies



D D*

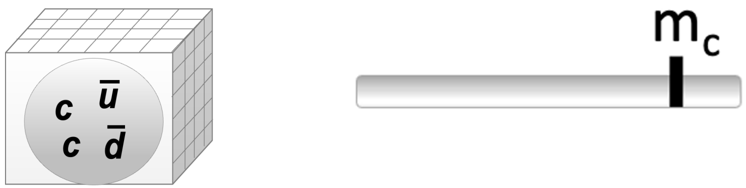
$$\mathcal{O} = (\bar{u}\gamma_5 c)_{\vec{p}_1}\ (\bar{d}\gamma_i c)_{\vec{p}_2} - (\vec{p}_1 \leftrightarrow \vec{p}_2) \qquad \vec{p}_{1,2} = \vec{n}_{1,2}\ \tfrac{2\pi}{L}$$

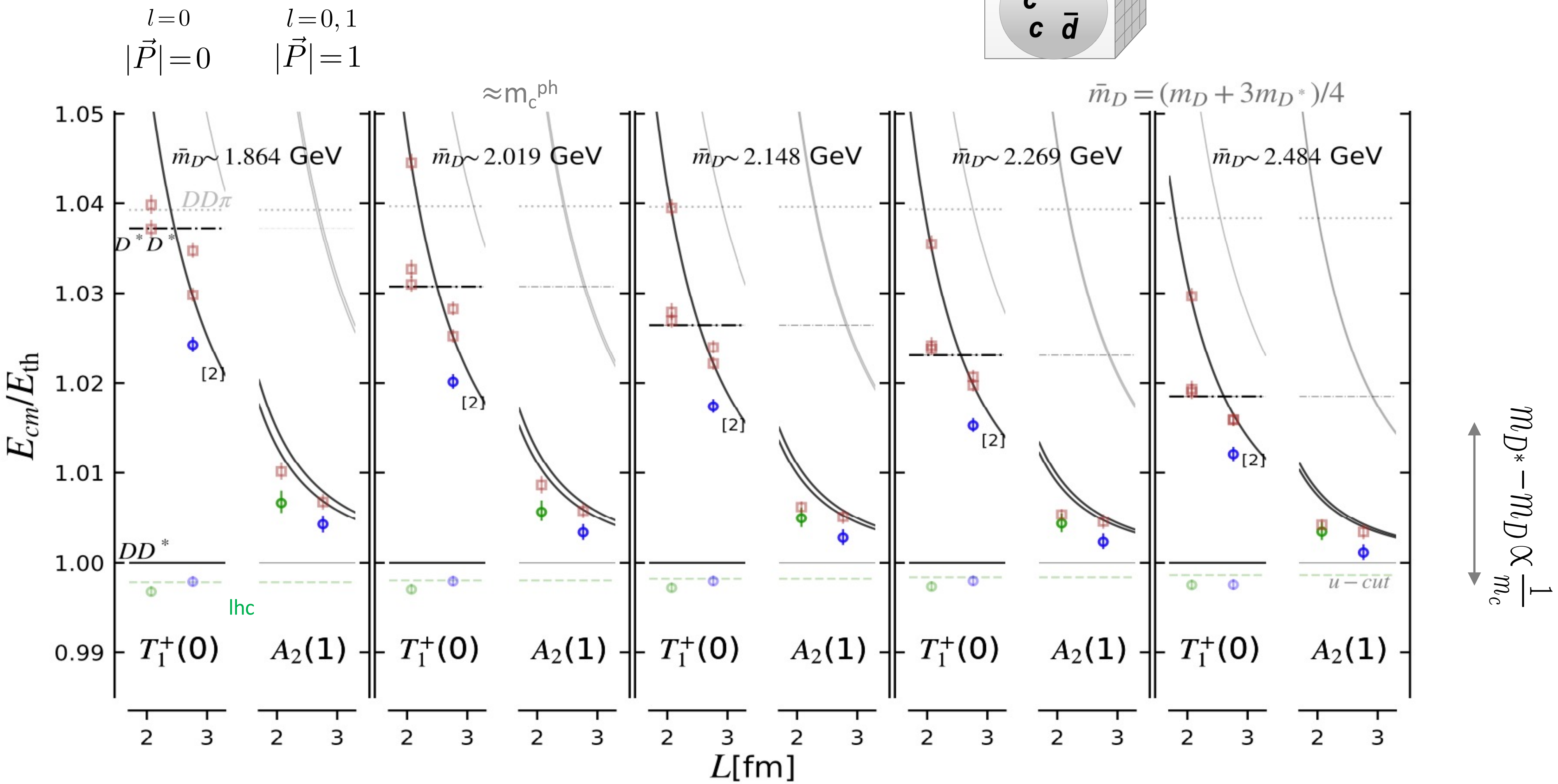
$$(\bar{u}\gamma_5\gamma_t c)_{\vec{p}_1}\ (\bar{d}\gamma_i\gamma_t c)_{\vec{p}_2}$$

recent Hsc 2405.15741
presented at the end

first extraction of T(E) for Tcc:
Padmanath & SP, 2202.10110 PRL

T_{cc} : finite-volume eigen-energies



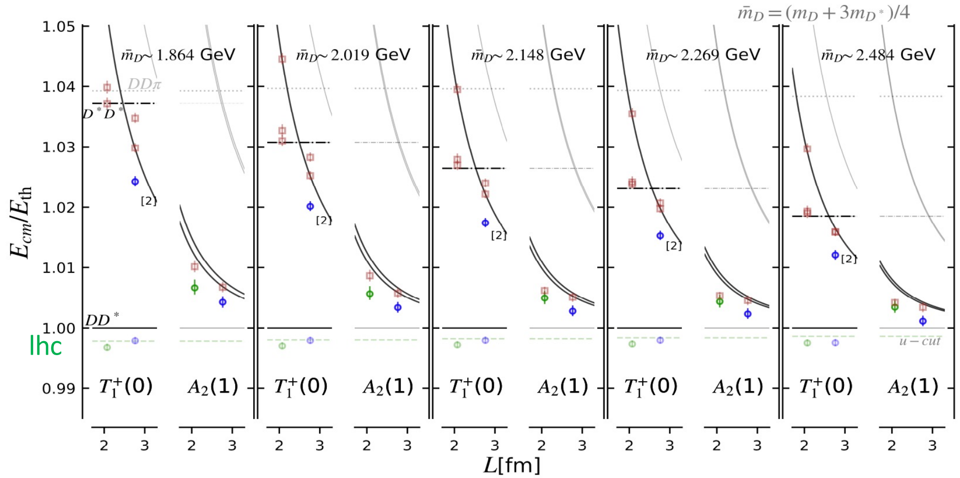


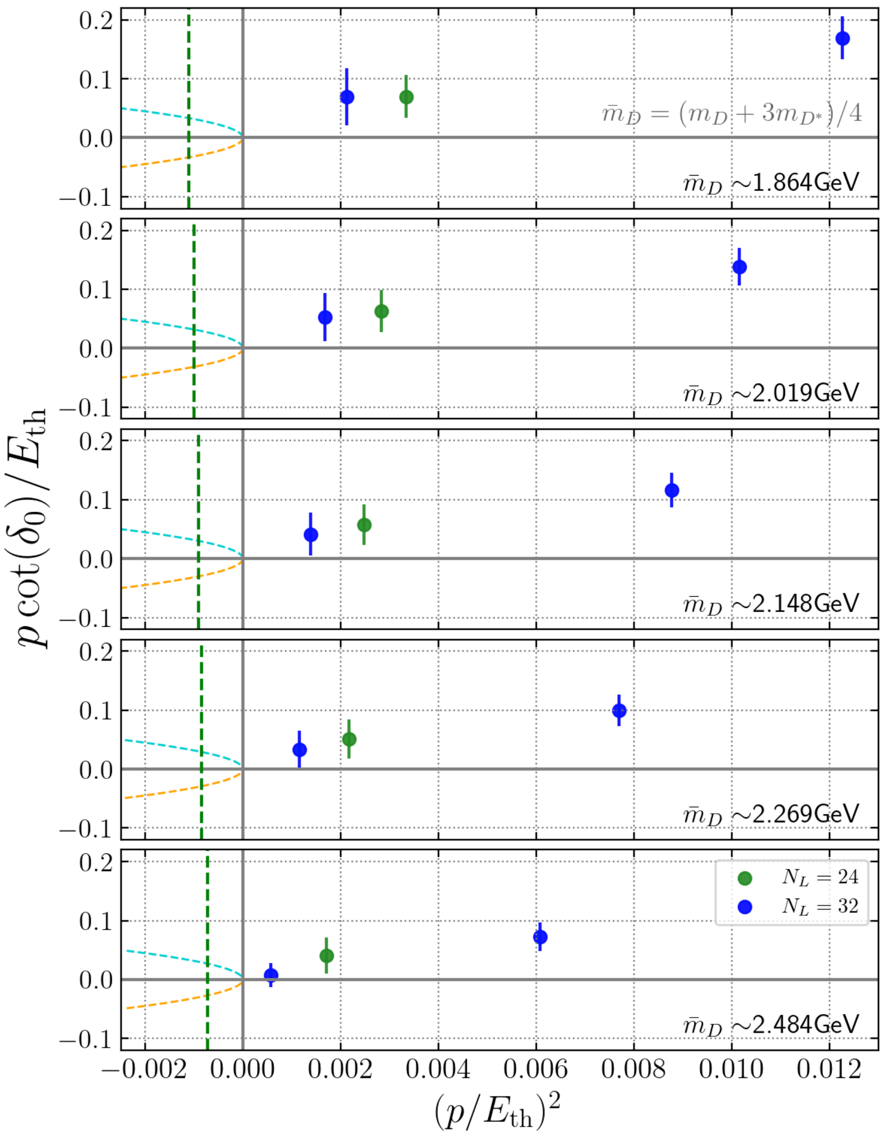
lines

$$E^{n.i.}=\sqrt{m_D^2+\vec{p}_1^{\,2}}+\sqrt{m_{D^*}^2+\vec{p}_2^{\,2}}$$
$$\vec{p}_i=\vec{n}_i\frac{2\pi}{L}$$

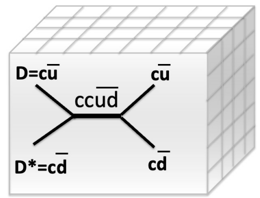
Collins, Nefediev, Padmanath , SP, 2402.14715, PRD

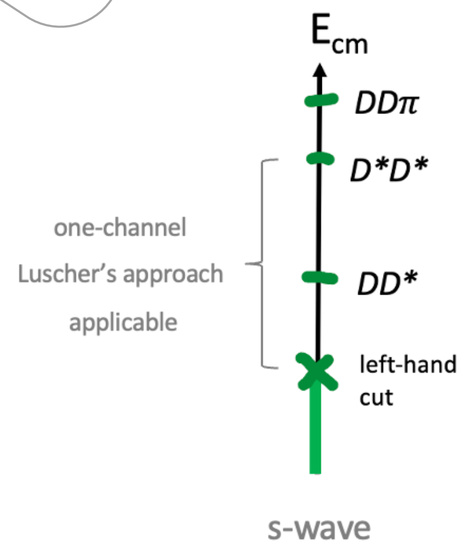
T_{cc} : scattering amplitude

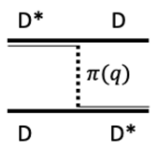




δ(E) E



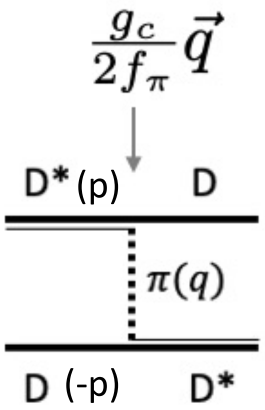




T_{cc} : Pion exchange, left-hand cut etc

$$q^2 = q_0^2 - \vec{q}^{\,2} \simeq (m_{D^*} - m_D)^2 - \vec{q}^{\,2}$$

Heavy meson ChPT



$$V_\pi^{cent}(\vec{q}) = \frac{g_c^2}{4f_\pi^2}\,\frac{\vec{q}^{\,2}}{q^2 - m_\pi^2} = \frac{g_c^2}{4f_\pi^2}\left(-1 + \frac{\mu_\pi^2}{\vec{q}^{\,2} + \mu_\pi^2}\right)$$

$$\mu_\pi^2 = m_\pi^2 - (m_{D^*} - m_D)^2$$

lat : $\mu_\pi^2 > 0$

ph : $\mu_\pi^2 < 0$

attraction at short distance: $-\delta^{(3)}(\vec{r})$

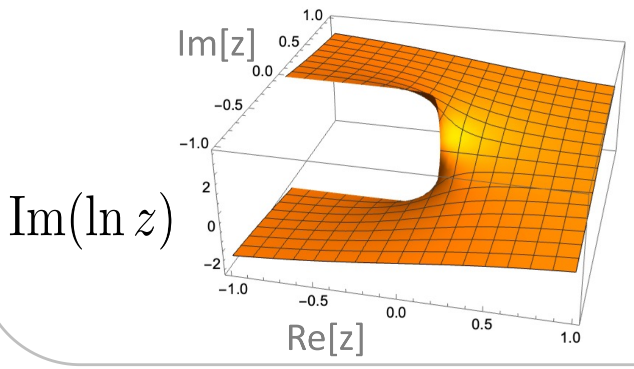
slight repulsion at long distance: $\frac{\mu_\pi^2}{r} e^{-\mu_\pi r}$

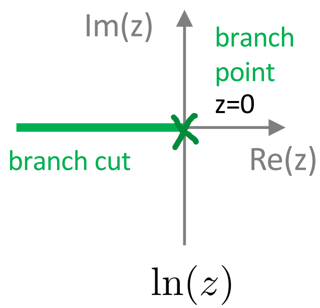
s-wave projection

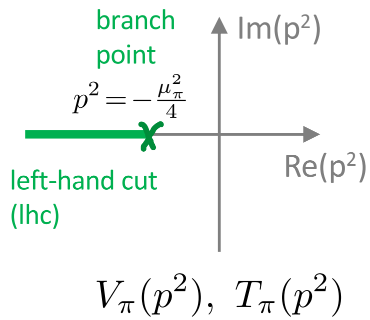
$$V_\pi^S(p,p) \propto \int V_\pi(\vec{q})\; d\cos\theta, \qquad \vec{q}^{\,2} = 2p^2(1 - \cos\theta)$$

$$V_\pi^S(p,p) \propto \ln\left(1 + \frac{4p^2}{\mu_\pi^2}\right)$$

complex p cot δ (Luscher's eq would render it real)







lhc slightly below DD*, BB*, NN ... th.

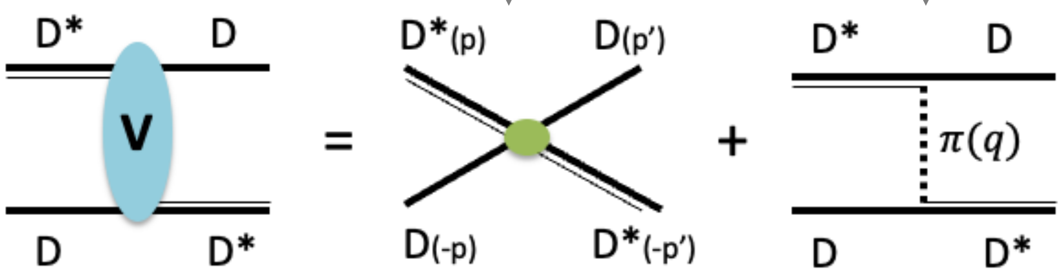
T_{cc} analysis based on EFT

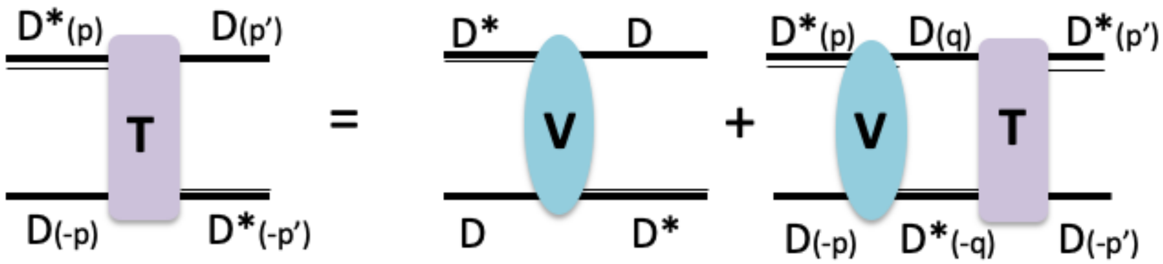
Collins, Nefediev, Padmanath , SP, 2402.14715, PRD

$c_{0,2}$ fitted from lat. data

significant short-distance attraction

$$V_{CT} = 2c_0 + 2c_2(p^2 + p'^2) \qquad\qquad \frac{g_c}{2f_\pi}\vec{q}$$





$$T = V - VGT$$

$$T = \frac{1}{V^{-1} + G}$$

$$T(\boldsymbol{p},\boldsymbol{p}';E) = V(\boldsymbol{p},\boldsymbol{p}') - \int \frac{d^3q}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q}) G(\boldsymbol{q};E) T(\boldsymbol{q},\boldsymbol{p}';E)$$

integral equation

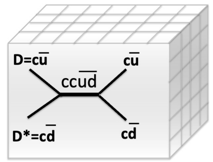
Limann-Schwinger eq.

Bethe-Salpeter eq.

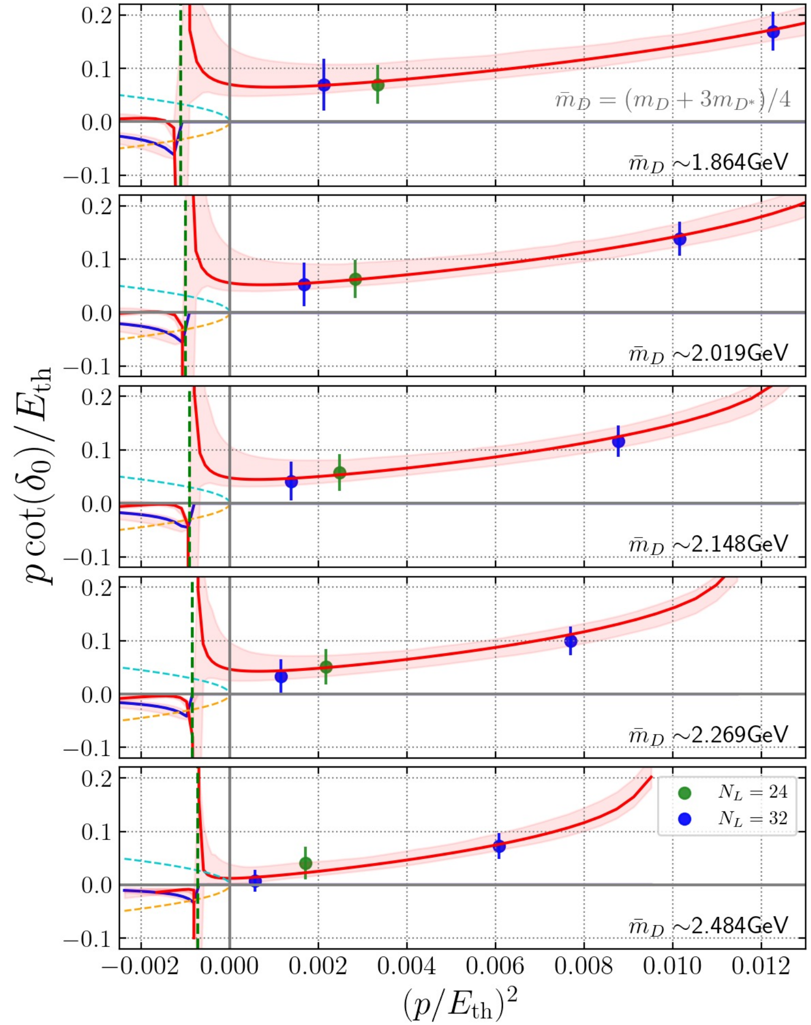
inspired by

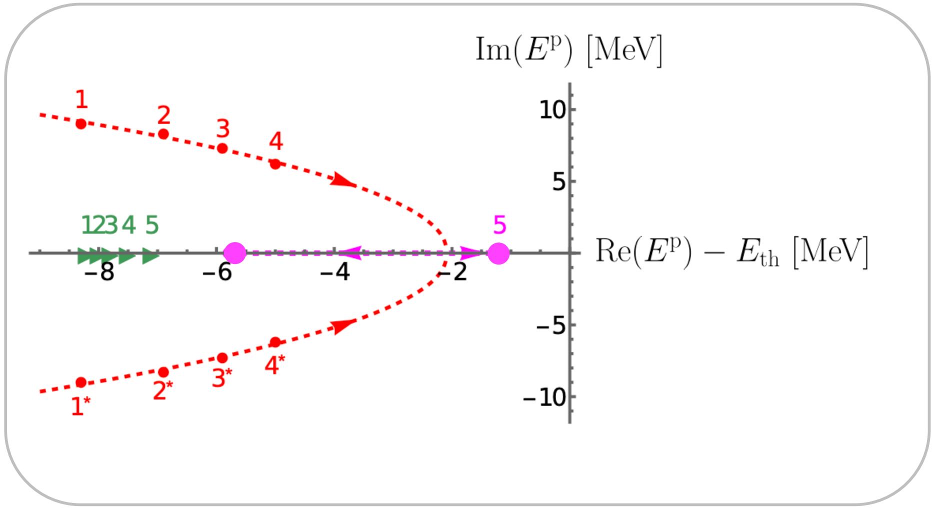
Du, Hanhart, Guo, Nefediev, Filin, et al, PRL 2023, 2303.09441

T_{cc} : scattering amplitude and pole trajectory

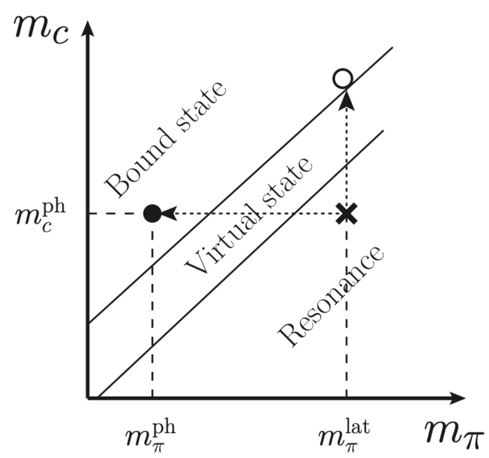


$m_\pi \simeq 280$ MeV





resonance pole
virtual state pole
lhc
arrow: increasing mc



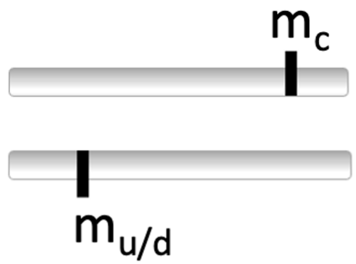
levels below lhc omitted from the fit

reassuring: plane-wave method incorporates levels below lhc and gets consistent s-wave amplitude Meng, Baru, Epelbaum et al., 2312.01930, PRD

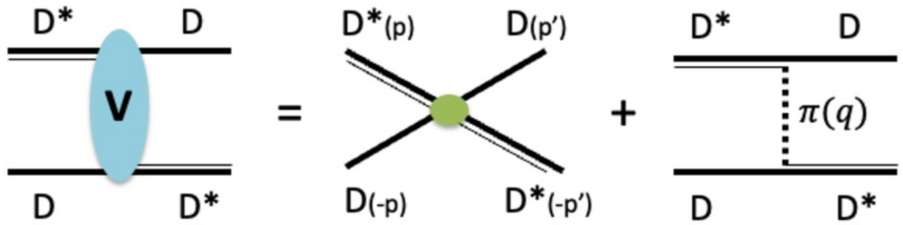
Collins, Nefediev, Padmanath , SP, 2402.14715, PRD

T_{cc} : interpretation

Collins, Nefediev, Padmanath , SP, 2402.14715, PRD

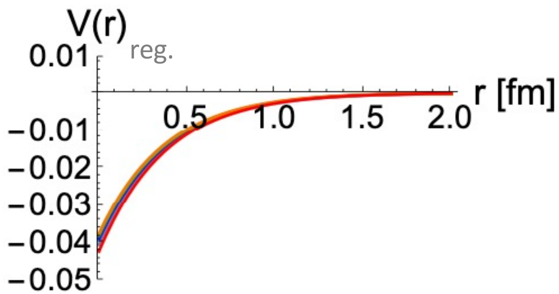


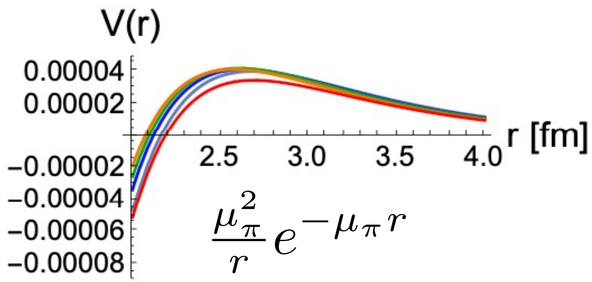
V(q)



FT

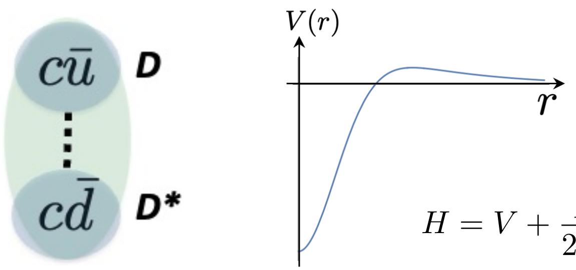
Potential for five different m_c





$$\frac{\mu_\pi^2}{r} e^{-\mu_\pi r}$$

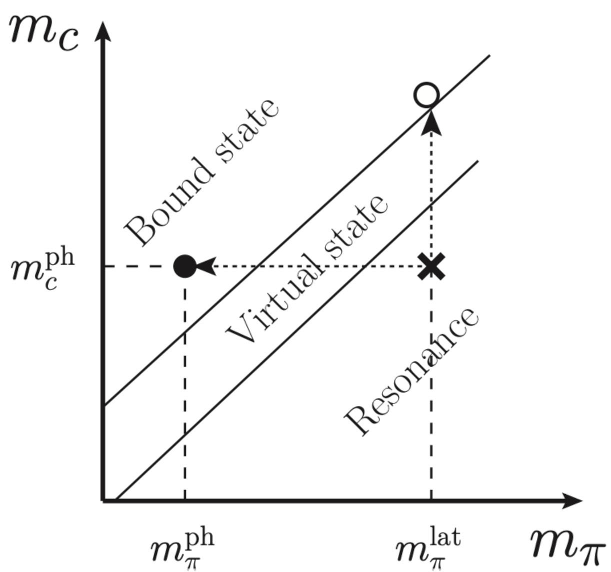
$\mu_\pi^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0$

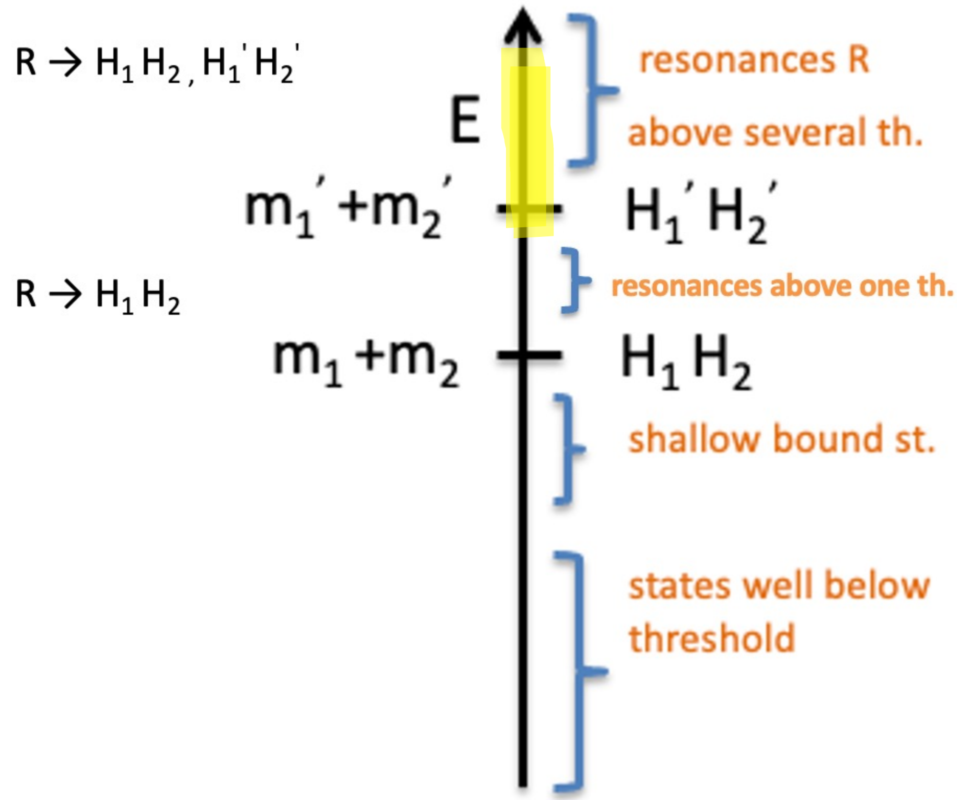


$$H = V + \frac{p^2}{2m_r}$$

V almost independent on m_c

attraction increases with decreasing $m_{u/d}$





Hadrons from coupled-channel scattering

Coupled-channel scattering

most of hadronic resonances decay strongly to several final states

$f_0(980) \to \pi\pi, K\bar{K}$

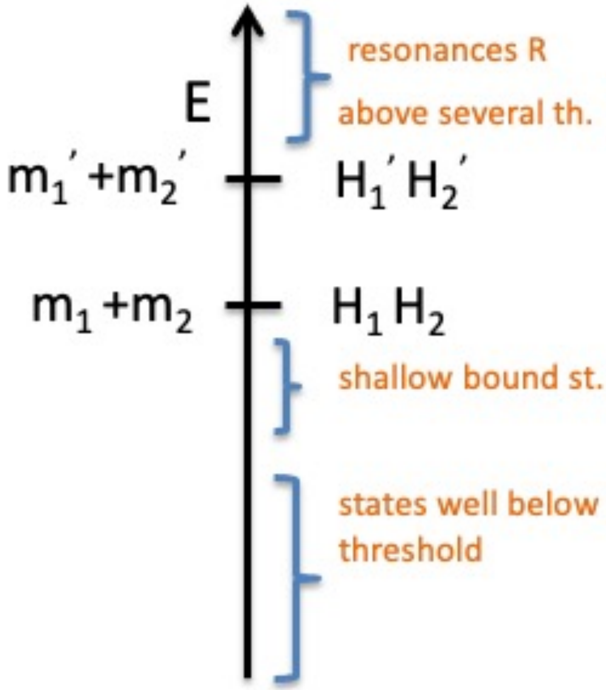
$a_0(980) \to \pi\eta, K\bar{K}$

$a_1(1260) \to \rho\pi, \sigma\pi, \ldots$

$K_0^*(1430) \to K\pi, K\eta, K\eta'$

$D_3^*(2750) \to D\pi, D^*\pi$

$R \to H_1 H_2, H_1' H_2'$



almost all exotic hadrons decay stronly to several final states

$\bar{c}cu\bar{d}: Z_c \to J/\psi\,\pi, D\bar{D}^*, \eta_c\rho, \ldots$

$\bar{b}bu\bar{d}: Z_b \to \Upsilon(1S)\pi, h_b(1P)\pi, B\bar{B}^*, \ldots$

$\bar{c}cuud: P_c \to J/\psi\, p, \Sigma_c D, \ldots$

$\bar{c}c\bar{c}c: X(6900) \to J/\psi\, J/\psi, \eta_c\eta_c, \ldots$

Resonances in Kπ, Kη coupled-channel scattering, I=1/2, l=0

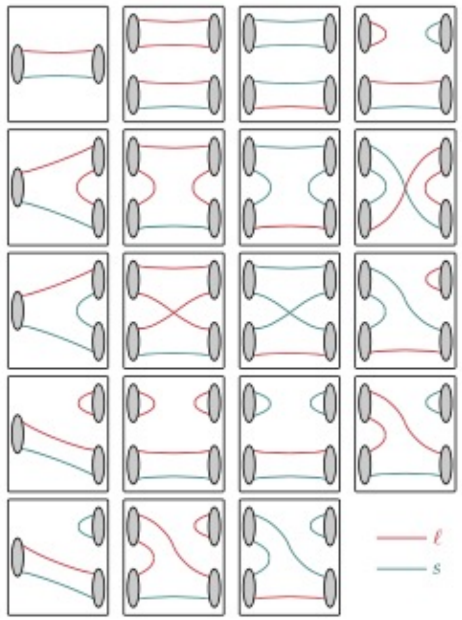
$\bar{s}u$

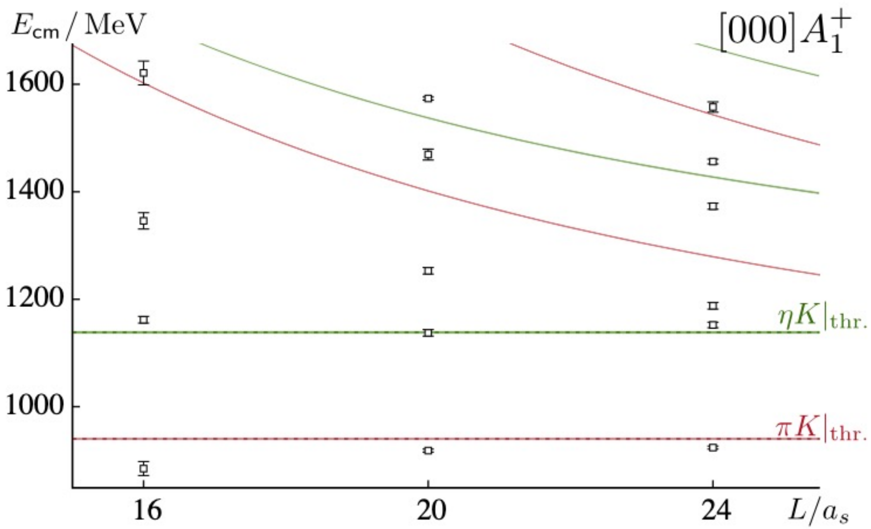
first coupled-channel scattering study:

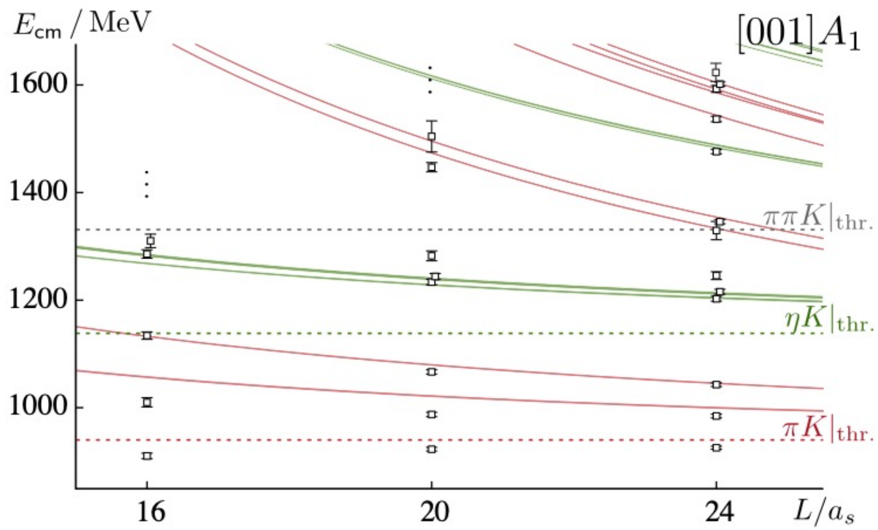
HSC (Wilson, Dudek,Edwards, Thomas): PRL 2014, PRD 2014

$$O : \bar{s}u, K(\vec{p}_1)\pi(\vec{p}_2), K(\vec{p}_1)\eta(\vec{p}_2)$$

Wick contractions







Sasa Prelovsek

Coupled-channel scattering matrix

Consider irrep where only partial wave l contributes (for simplicity)

one-channel scattering

1x1 1x1 1x1

$$S = I + i\frac{2p}{8\pi E}\mathcal{M}$$

$\mathcal{M}$: $K\pi \to K\pi$

M(E)

two-channel scattering

2x2 2x2 2x2

$$S = I + i\frac{2p}{8\pi E}\mathcal{M}$$

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12} & \mathcal{M}_{22} \end{pmatrix} = \begin{pmatrix} K\pi \to K\pi & K\pi \to K\eta \\ K\eta \to K\pi & K\eta \to K\eta \end{pmatrix}$$

M_{ij} (E)

Sasa Prelovsek

Determination of coupled-channel scattering matrix for 2 channels

Consider irrep where only partial wave l contributes (for simplicity); $E=E_{cm}$

E=lattice eigen-energy $$\det[\mathcal{M}^{-1}(E) - iF(E)] = 0$$

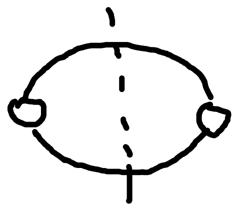
quantization condition

Sharpe & Hansen 1204.0826 and others

$$S = I + i\frac{2p}{8\pi E}\mathcal{M}$$

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12} & \mathcal{M}_{22} \end{pmatrix}$$

$$F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}$$



F comes from bubble diagram:

diagonal in channel space

each element same as before

$E=E^{lat}$ for given P, irrep $$f[\mathcal{M}_{11}(E), \mathcal{M}_{22}(E), \mathcal{M}_{12}(E)] = 0$$

impossible to determine

all three from one equation

rescue: parametrization of M(E)

general idea suggested by

Doring, Meissner, Oset, Rusetsky 1205.4838

$E=E_{cm}$

$$\det[\mathcal{M}^{-1}(E)+iF(E)]=0$$

quantization condition

M is diagonal for

scat of parrticles

with S=0

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \mathcal{M}_{12} & \mathcal{M}_{22}\end{pmatrix} \qquad F=\begin{pmatrix}F_{11} & 0\\ 0 & F_{22}\end{pmatrix}$$

1. parametrize M_{ij} as a function of E or p via some parameters (C)

   simple example $M_{ij}(E)=A_{ij}+B_{ij}\,E^2$

$M_{ij}(E)=M_{ij}(E,C)$

2. parameters C chosen such that det[]=0 at $E=E_n^{lat}$

   (hard to satisfy exactly)

$$\det[\mathcal{M}^{-1}(E,C_l^i)+i\ F(E)]_{L,\vec{P},\Lambda}=0$$

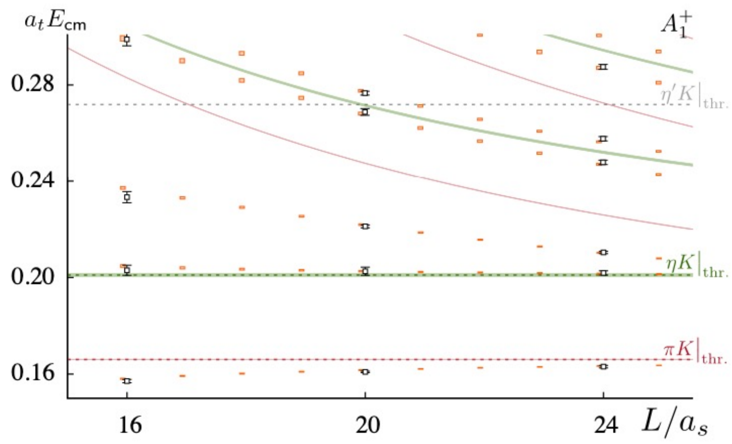
for all L,P, Λ=irrep

3. in practice: find C that satisfy det[]=0 best by minimizing chi2 below

$$\det[\mathcal{M}^{-1}(E,C_l^i)+i\ F(E)]_{L,\vec{P},\Lambda}=0 \rightarrow E_n^{model}(C)$$

$$\chi^2(C)=\sum_{a,b}\,[E_a^{lat}-E_a^{model}(C)]\ \mathrm{cov}_{ab}^{-1}\ [E_b^{lat}-E_b^{model}(C)]$$

$\circ\, E^{lat}$ $\circ\, E^{model}$



sum a,b over all discrete energy levels n and all L,P, Λ=irrep studied

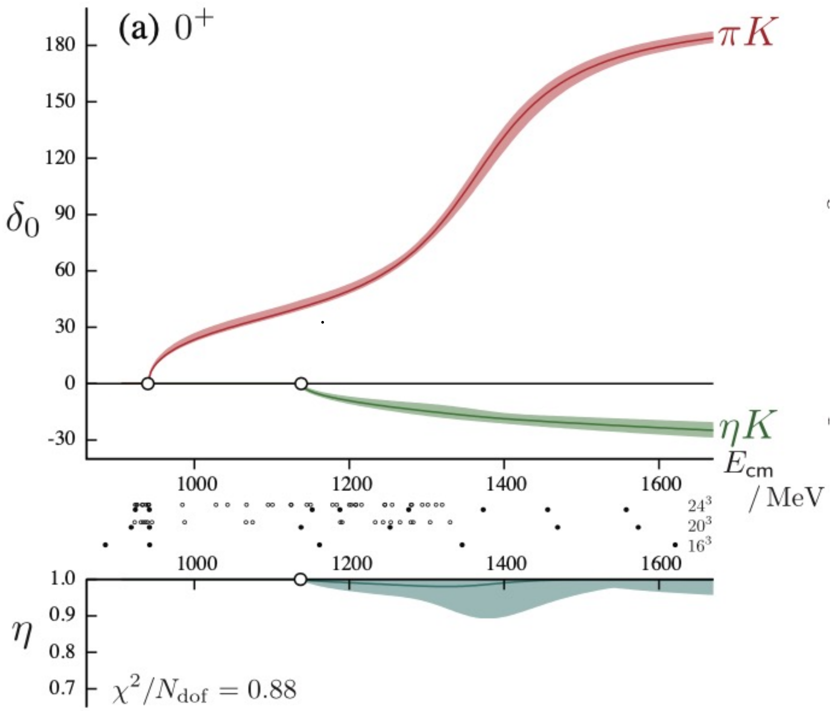
Results for $K\pi$, $K\eta$ scattering in I=1/2, l=0

$\bar{s}u$

HSC (Wilson, Dudek,Edwards, Thomas):
PRL 2014, PRD 2014

$$
\begin{array}{cccc}
 & K\pi & K\eta & \\
S = & \eta\, e^{2i\delta_1} & \sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & K\pi \\
 & \sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & \eta\, e^{2i\delta_2} & K\eta
\end{array}
$$

eta=1 : decoupled channels



these two channels are almost decoupeld

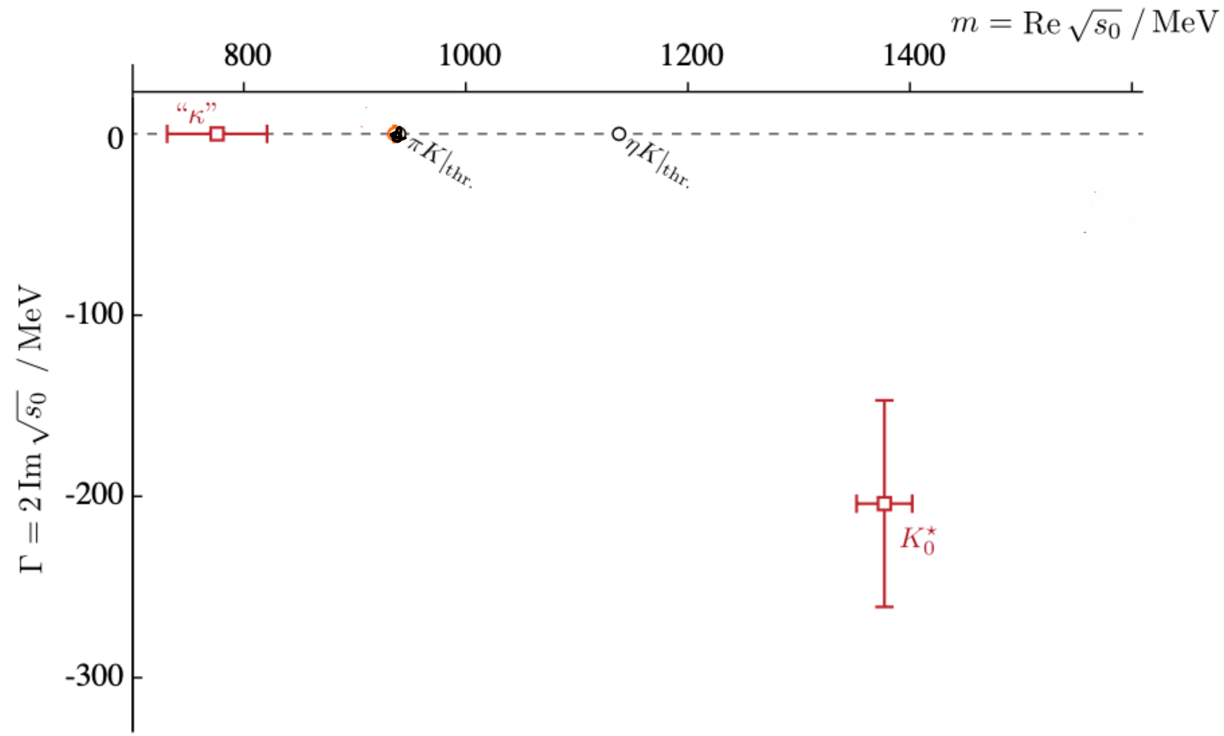
for examples of channels that are not decoupeld

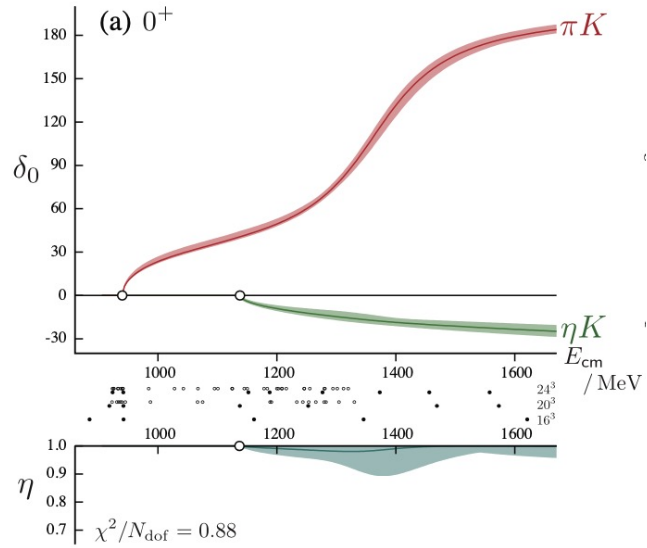
see further works by HSC

Location of poles in Kπ, Kη scattering: I=1/2, l=0

$\bar{s}u$

HSC (Wilson, Dudek,Edwards, Thomas):
PRL 2014, PRD 2014





Charmonium(like) resonances and bound states

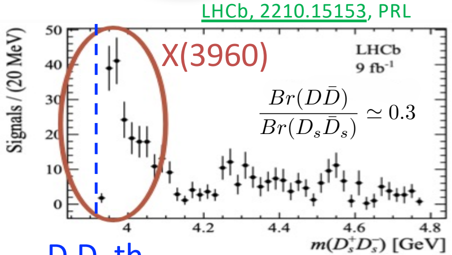
$\bar{c}c$, $\bar{c}q\bar{q}c$ q=u,d,s $I=0$

$\bar{D}_s D_s$ J^P=0$^+$ state

$$T_{ij}(E_{cm}) \sim \frac{c_i\ c_j}{E_{cm}^2 - m^2}$$

$$\frac{|c^2_{D\bar{D}}|}{|c^2_{D_s\bar{D}_s}|} \overset{\text{lat}}{=} 0.02\ ^{+0.02}_{-0.01}$$

LHCb, 2210.15153, PRL



D_sD_s th.

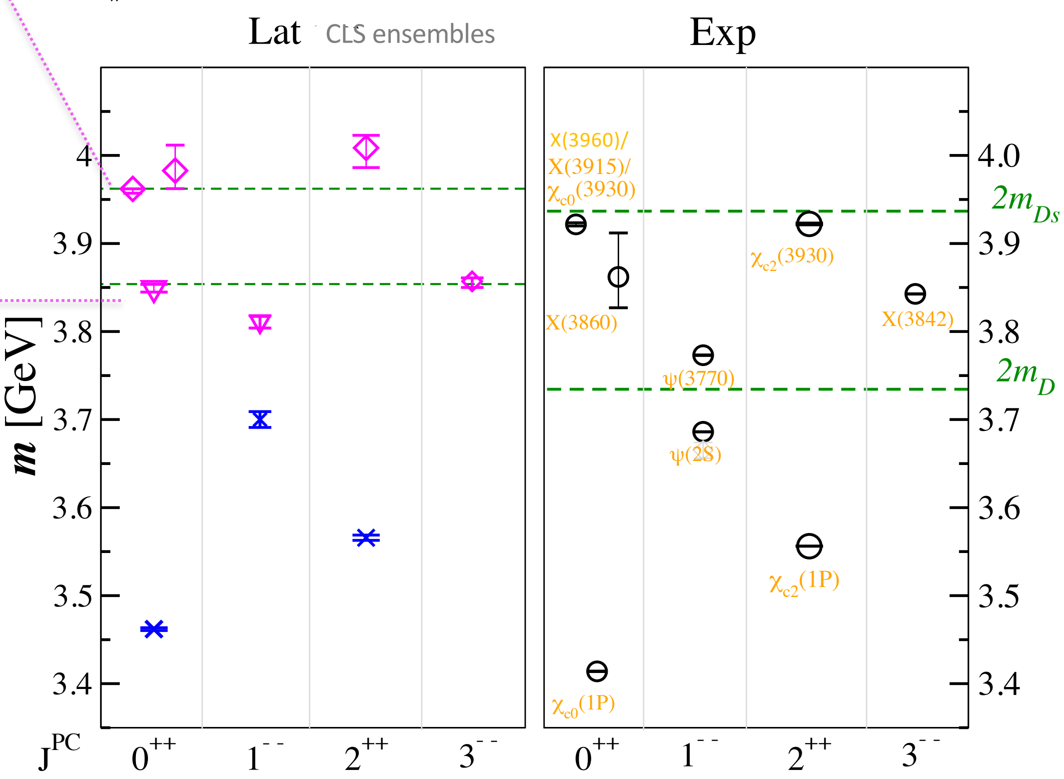
$\bar{D}D$ J^P=0$^+$ state

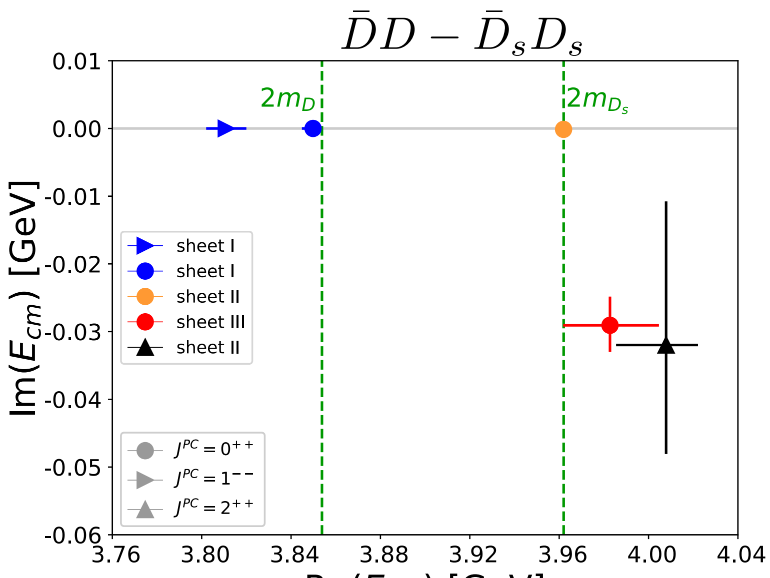
predicted in models [Oset et al, 0612179 PRD, Guo et al 2101.01021]

seen in re-analysis of exp.
[Danilkin et al 2111.15033,
Ji, F.K. Guo et al., 2212.00613]

+ expected conventional charmonia

$m_\pi \simeq 280$ MeV $m_D \simeq 1927$ MeV





S.P. , Collins, Padmanath, Mohler, Piemonte
2011.02541 JHEP, 1905.03506 PRD

Likely interpretation of some near-threshold states: “molecules” attracted by V exchange

a number of pheno studies
Oset et al, 0612179 PRD,
Guo et al, 2101.01021,...

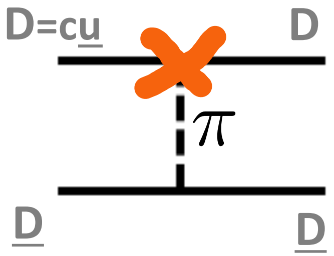
$\bar{c}q\bar{q}c$ $I=0$ $J^P=0^+$

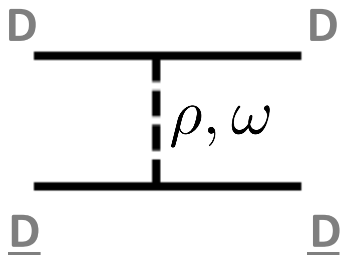
now support also from lattice

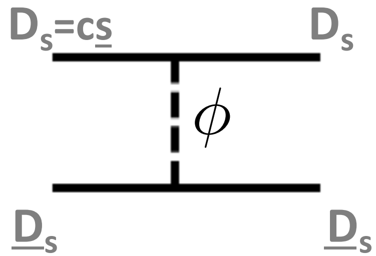
$\bar{D}D$

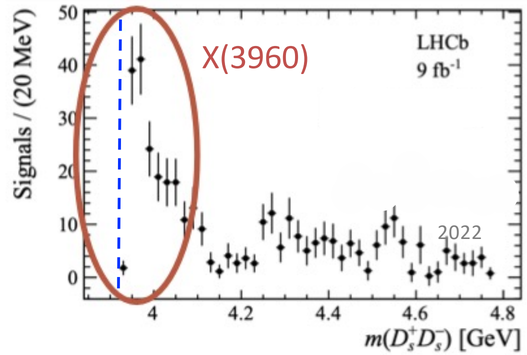
$\bar{D}_s D_s$

$\underline{D}_s D_s$ th.









Coupled-channel *DD*-D*D** scattering

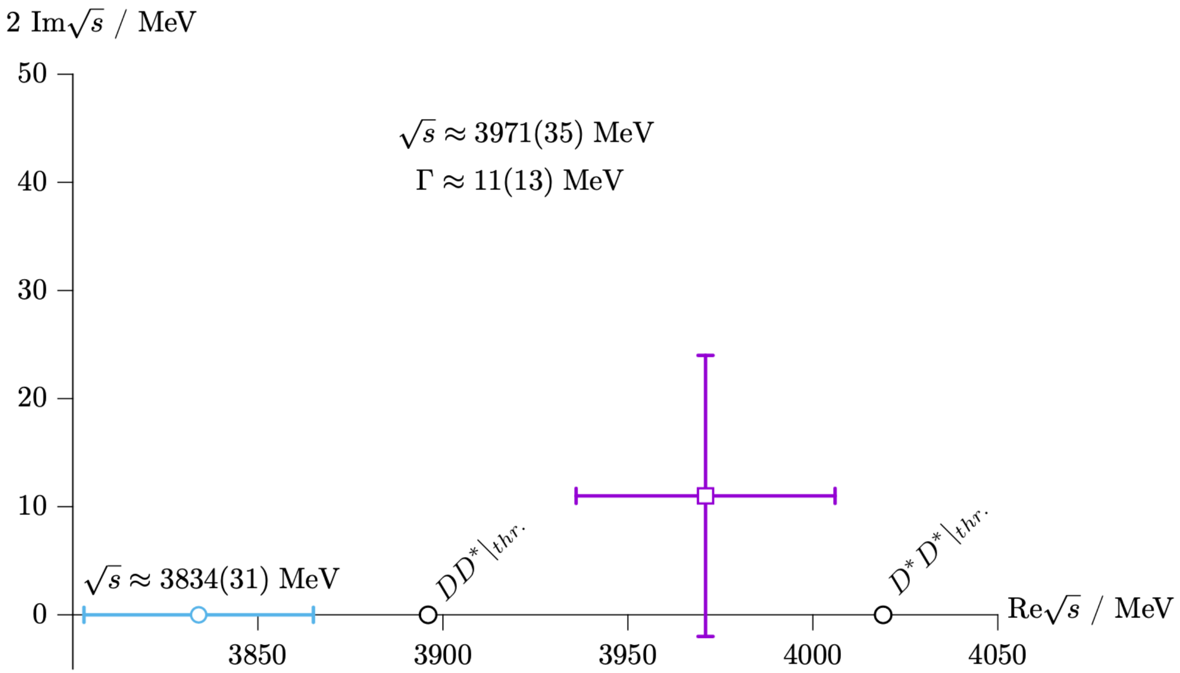
Hadspec 2405.15741

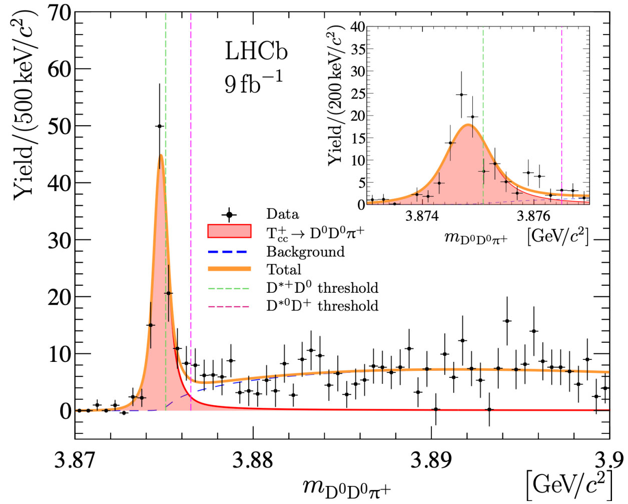
$$m_\pi \simeq 391 \text{ MeV}$$

T_{cc} virtual state below DD* threshold (effects from left-hand cut not incorporated)

+

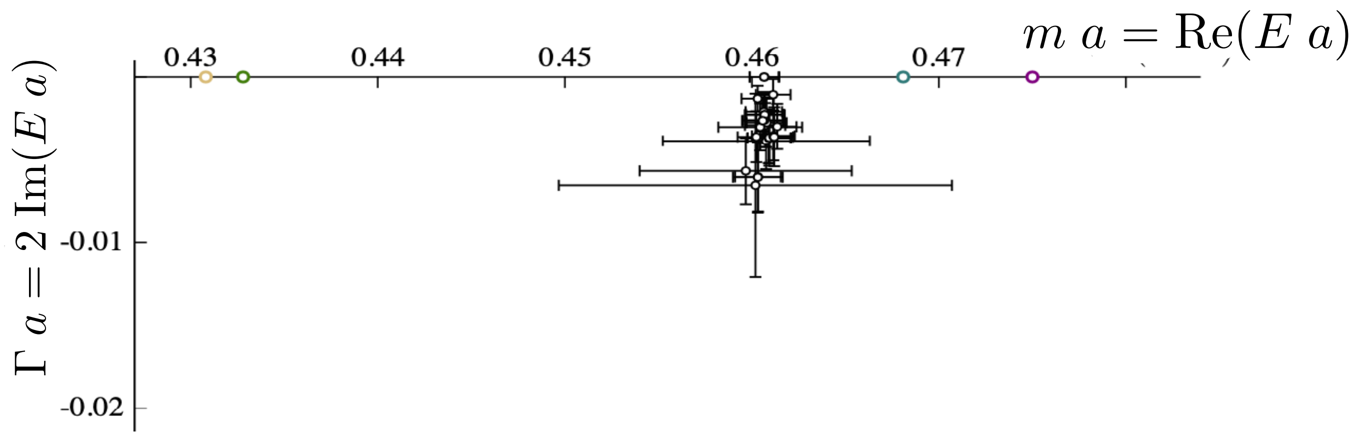
T_{cc}' resonance below D*D* threshold : look for it in experiment !



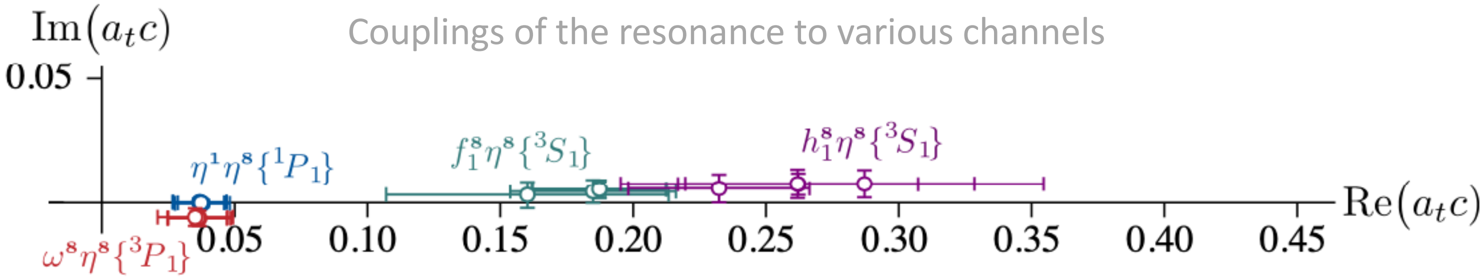


light hybrid meson π_1 from lattice

$\bar{d}Gu$ $J^{PC} = 1^{-+}$



Couplings of the resonance to various channels



$$T_{ij} \sim \frac{c_i\ c_j}{E_p^2 - E^2}$$

Woss et al. (HadSpec) 2009.10034

$m_u = m_d = m_s$, $m_\pi \approx 700\ MeV$

pheno analysis

physical world

$\rho\ \pi$ $\eta'\ \pi$ $f_1\ \pi$ $b_1\ \pi$

dominant coupling

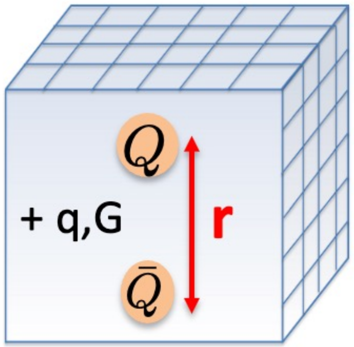
resemblence to experimental $\pi_1(1564)$: COMPASS+JPAC Rodas 1810.04171 [PRL]

$\pi_1(1564)$ in COMPASS+JPAC replaces two older resonances $\pi_1(1400)$ and $\pi_1(1600)$

$\pi_1(1600)$ PDG 2024

Exotic hadrons from static potentials

Static potentials from Born-Oppenheimer approximation



System with

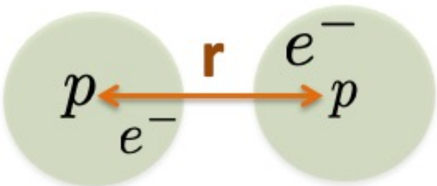
- two heavy partices QQ or QQ or ...
- light degrees of freedom q=u,d, G

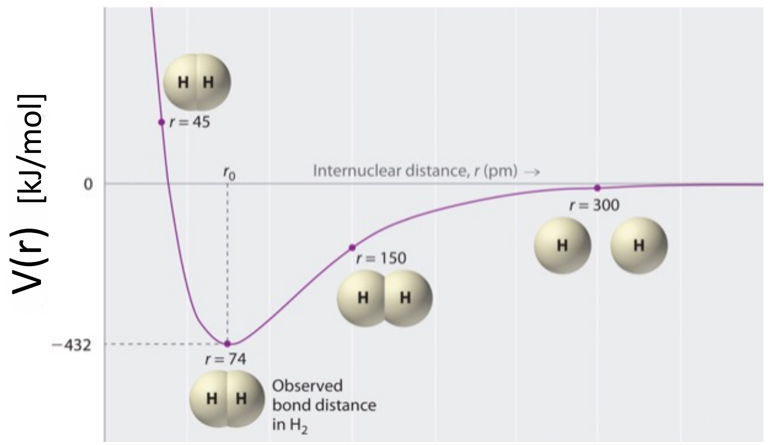
$$E = m_Q + m_{\bar{Q}} + W^Q_{kin} + W(q,G)$$

$$E = m_Q + m_{\bar{Q}} + W^Q_{kin} + V(r)$$

$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t)$$

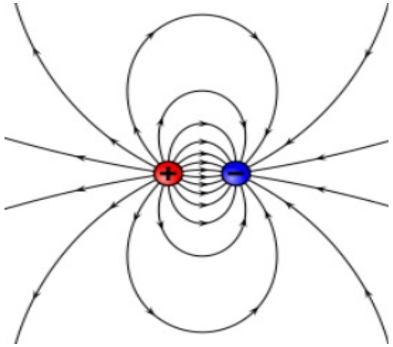
H_2

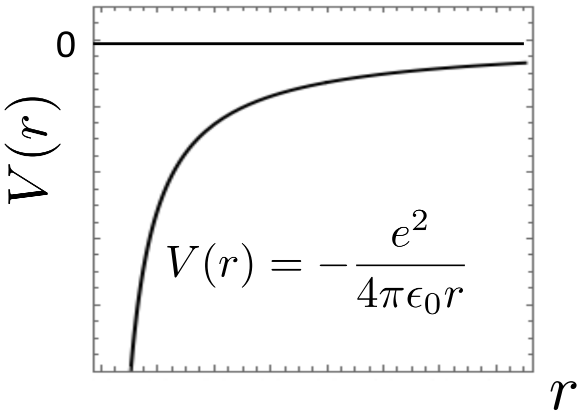




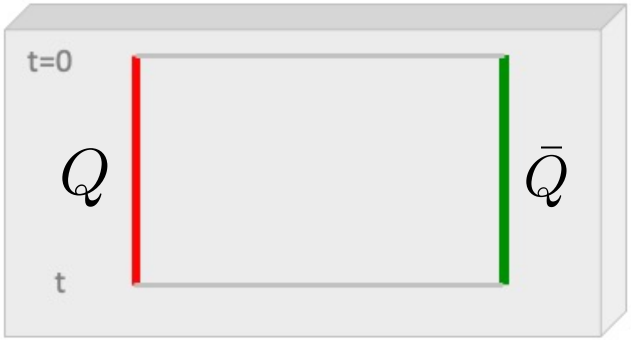
Potential and confinement

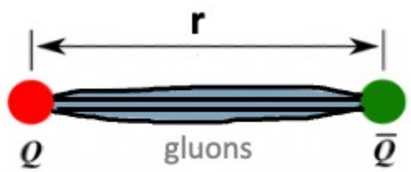
EM interactions

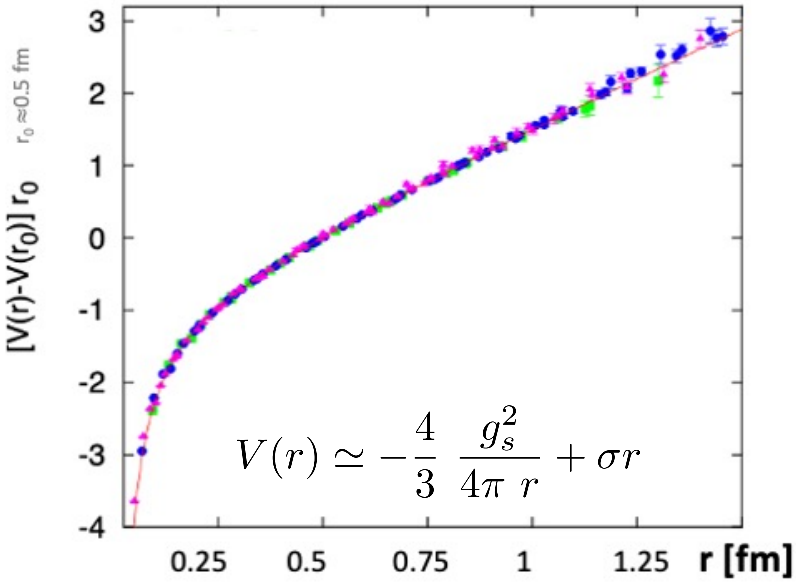




strong interactions (QCD without dynamical quarks)







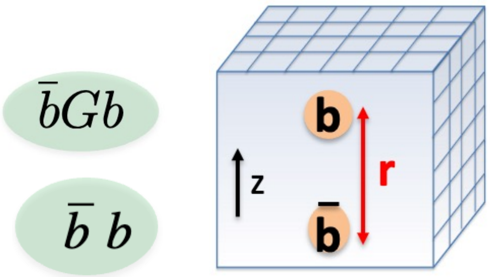
Bali, Phys. Rept., 2001

Bottomonia and hybrids

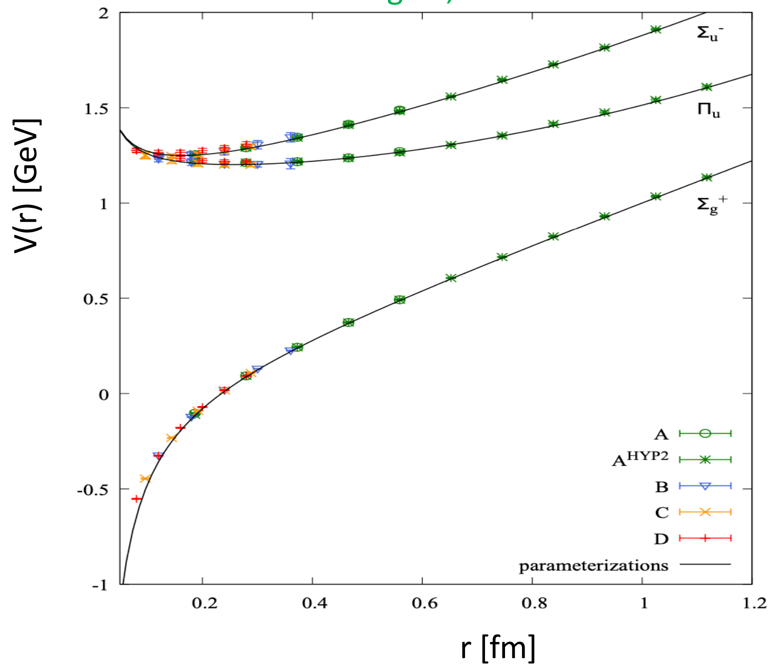
omit strong decays

quenched

I=0, varius J^P

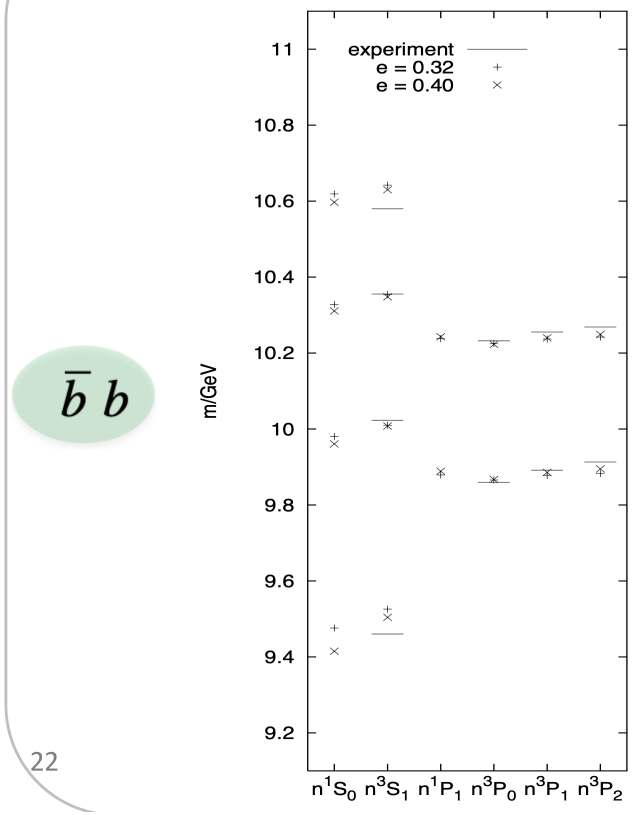


Schlosser & Wagner, 2111.00741



Bali, Phys. Rept., 2001

based on V from: Bali et al hep-lat/9703019

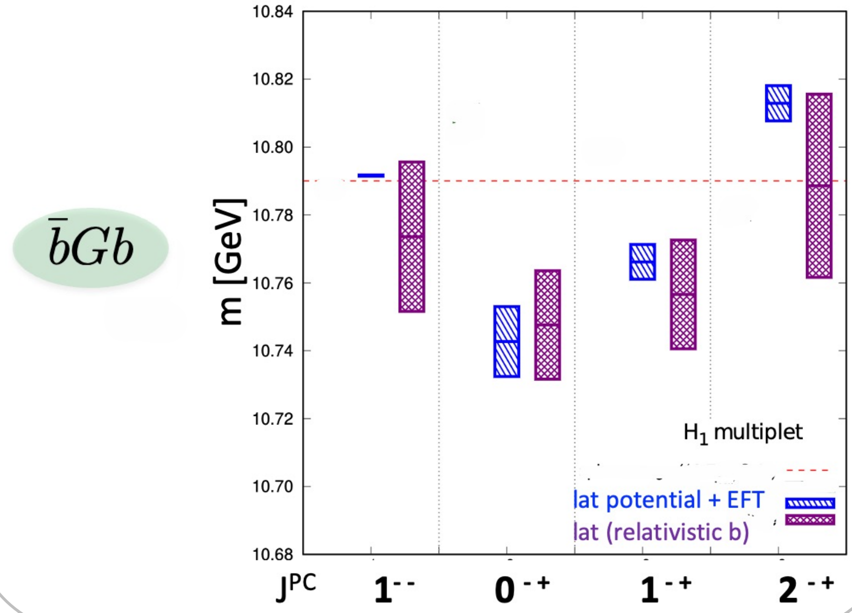


22

blue: Segovia, Tarrus; Brambilla @ MITP 2022

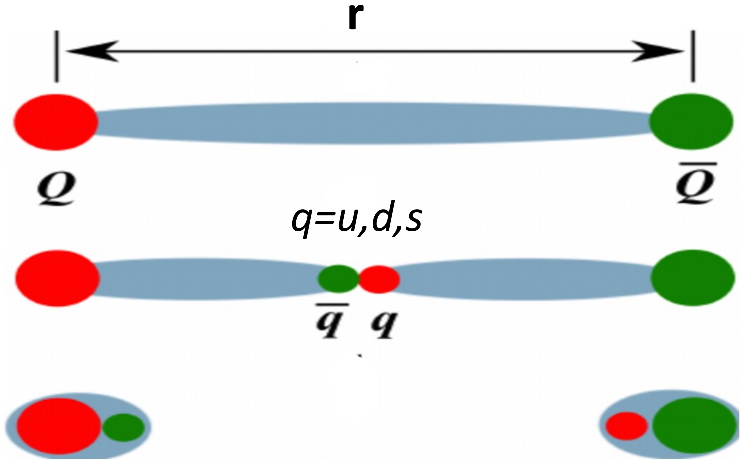
based n V from: Juge, Kuti, Morningstar, 1997, 1998

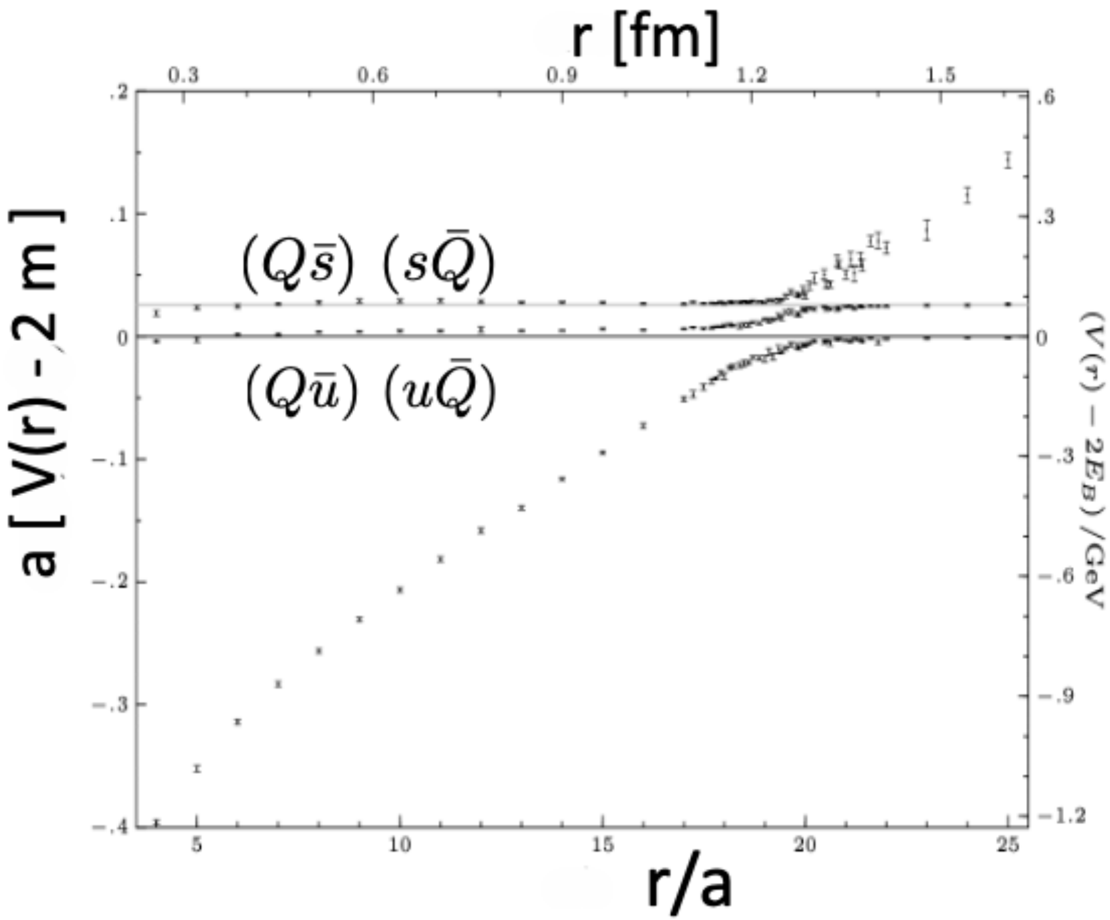
violet: Ryan & Willson 2020



Confinement and string breaking

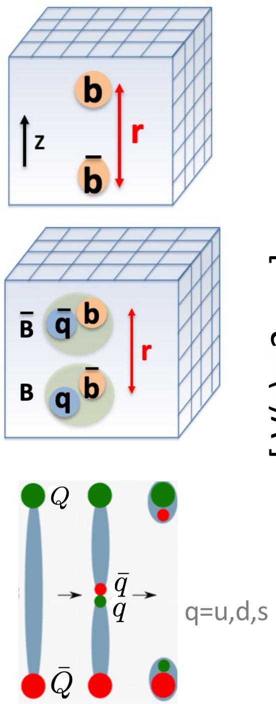
in QCD





Bulava et al, PLB793 (2019) 493

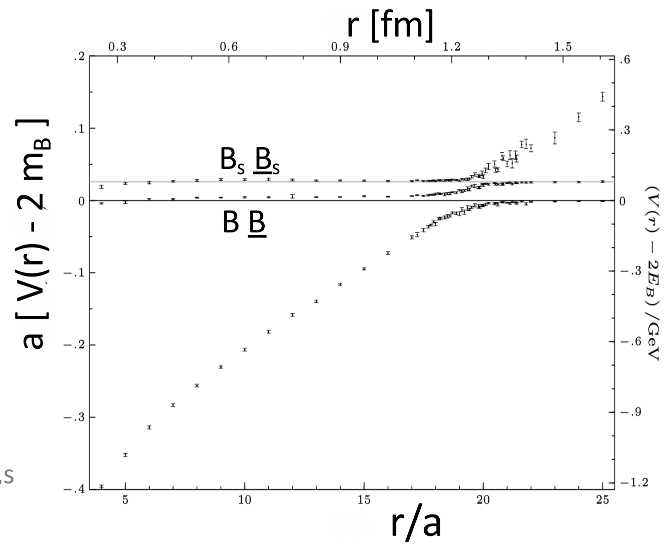
Bottomonia and bottomonium-like states (I=0)



string breaking

SESAM, Bali et al, 2005, 0505012

Bulava et al, 1902.04006, PLB

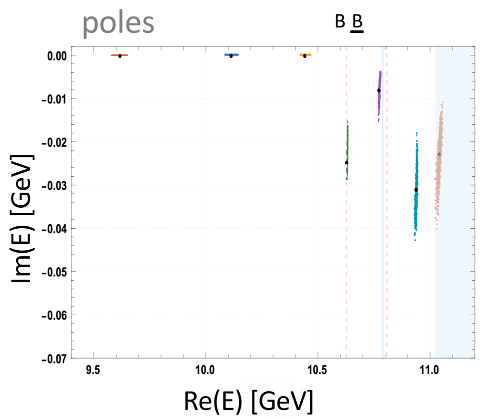


Bicudo, Cardoso, Mueller, Wagner, 2205.11475,

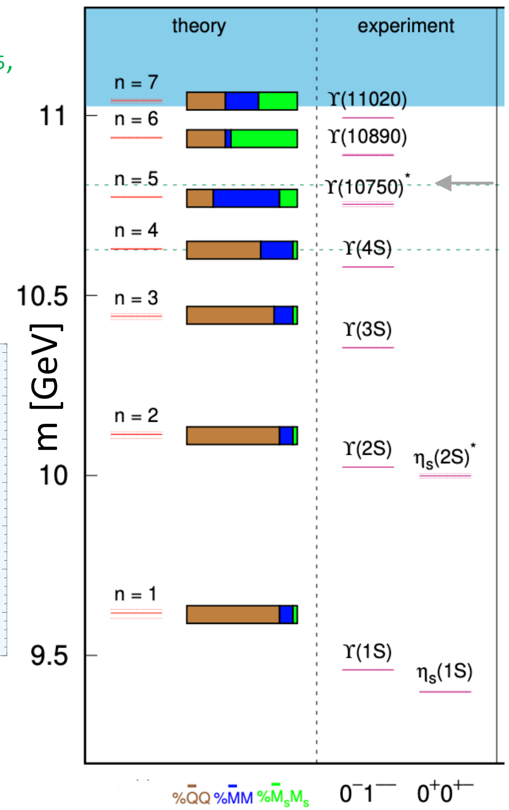
$\tilde{J}$=0: here; $\tilde{J}$=1,2,3 in the paper;

$\tilde{J}$=total ang. mom. without heavy quark spins

$\bar{b}b$ $\bar{b}b\bar{q}q$



see also: Castella, 2207.09365, 2207.09365



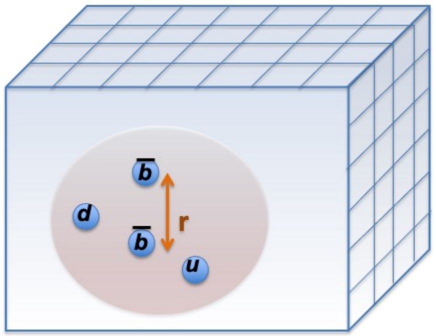
exotic st
[Belle 2019. talk by Zukhova]

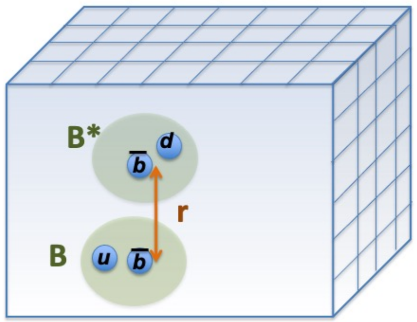
$\bar{b}\,\bar{b}\,d\,u$ Lattice QCD, static b quarks and Born-Oppenheimer : **done**

Lattice QCD

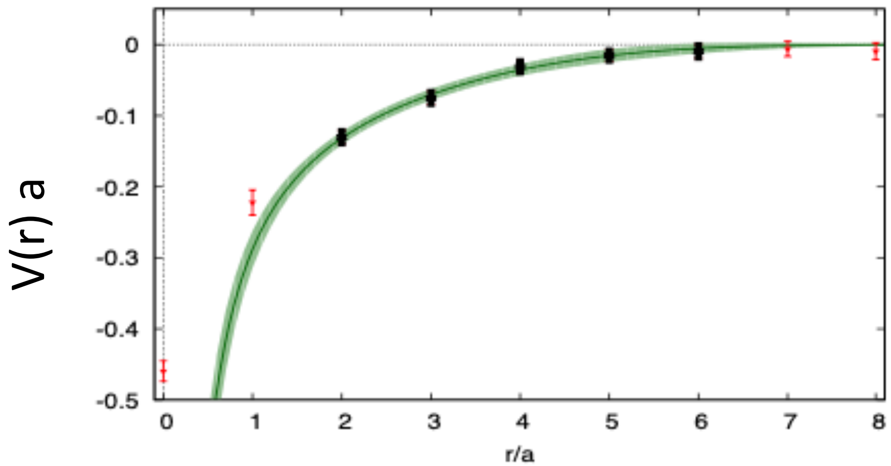
non-relativisic Schr. eq.

$$E_n(r) \rightarrow V(r) \rightarrow m$$



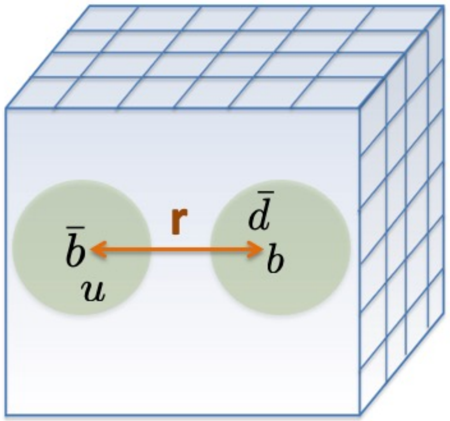


$m - m_B - m_{B*} = -\ 38\ (18)$ MeV



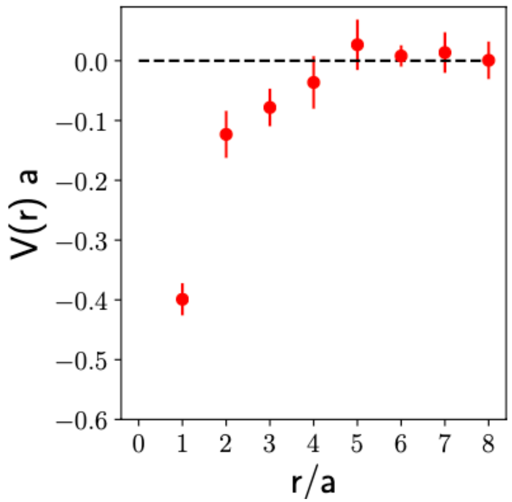
a number of works by Bicudo, Wagner, Peters, Cichy (above 1209.6274)

$\bar{b}b\bar{d}u$

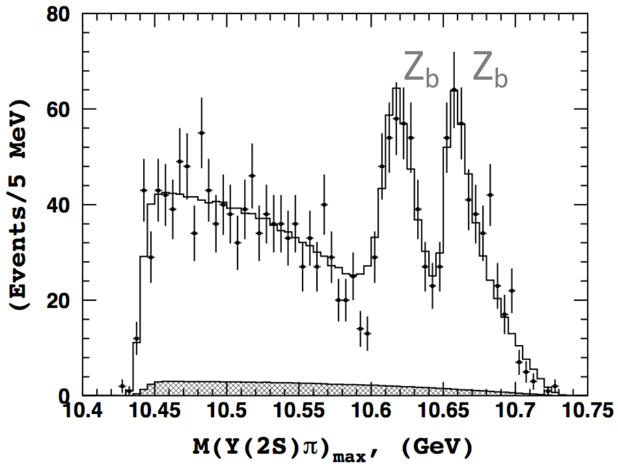


S.P., Bahtiyar, Petkovic, PLB 2019

Sadl, S.P., PRD 2021



Belle 2011 PRL 2011



challenge

$$\bar{b}\,b\,\bar{d}\,u \to \bar{b}b + \bar{d}u$$

$$\bar{b}\,b\,\bar{d}\,u \to \bar{b}u + \bar{d}b$$

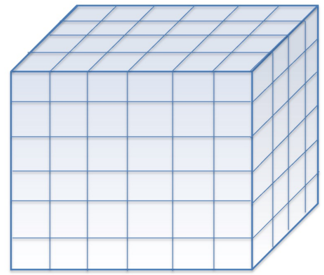
see many disclaimers in

the quoted papers;

still mostly open problem

All presented results are extracted from E_n (except from HALQCD Tcc)

$$\langle C \rangle = \int DG\, Dq\, D\bar{q}\, C\, e^{-S_{QCD}/\hbar}$$

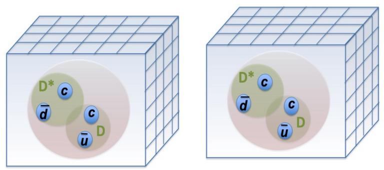


often "non-precision" studies:

single a, $m_{u/d} > m_{u/d}^{phy}$, $m_\pi > 140$ MeV

$$C_{ij}^{2pt}(t) = \langle 0 | \mathcal{O}_i(t)\ \mathcal{O}_j^+(0) | 0 \rangle = \sum_n \langle 0 | \mathcal{O}_i | n \rangle\ e^{-E_n t_E} \langle n | \mathcal{O}_j^+ | 0 \rangle$$

t, J^P, I   t=0, J^P, I



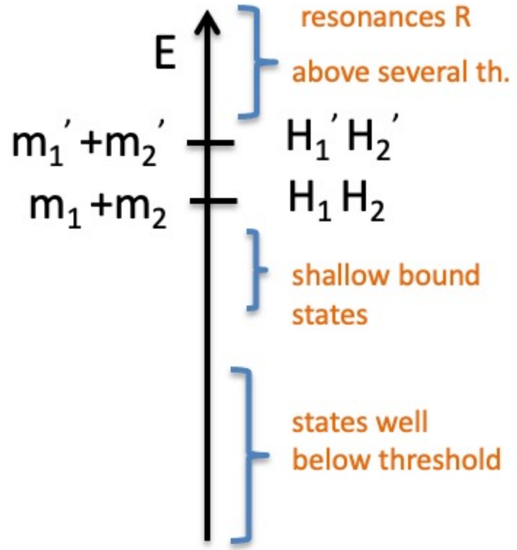
$$\mathcal{O} = \mathcal{O}(q, G)$$

- for strongly stable state well below threshold : ***E_n(P=0) = m***
- resonances (Luscher's relation) $\qquad E_n^{cm} \to T(E_n^{cm})$
- static potentials: $\qquad E_n \to V(r)$

Conclusions

great experimental results

Status on exotic hadrons from Lattice :



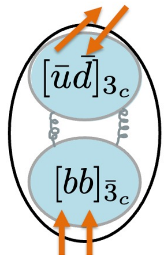
- exotic hadrons that are not resolved (yet)

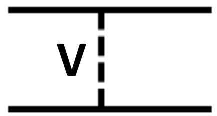
  strongly decay via many decay channels: $Z_c(4430)$, X(6900),...

- available: valuable results on exotic (and conventional) hadrons

  strongly stable ; strongly decaying to 1,2,3 channels

  support for specific binding mechanisms





...

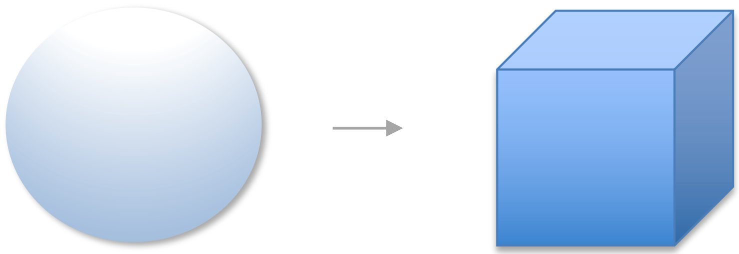
Pc, certain charmonium-like states

  one picture can not explain all exotic hadrons

  for each exotic hadron there is at least one viable picture

Backup

Brief intermezzo :
good quantum numbers for reduced rotational symmetry



Symmetries (for system with total momentum zero)

continuum

P: good

Rotations:

SO(3)

infinite number of elements $R = e^{i\vec{\varepsilon}\vec{J}}$

cubic lattice

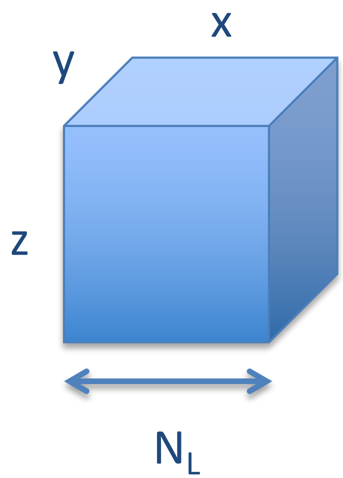
P: good

Rotations: cubic box periodic BC in x,y,z

Octahedral group O

24 elements

E	$8C_3$	$6C'_2$	$6C_4$	$3C_2 = (C_4)^2$



Irreducible representations under rotations

continuum

Rotations:

continuous SO(3)

infinite number of elements

irreducible rep.: spin J

dimension 2J+1

m_J=-J,..,J transform within themselves

spin J: good quantum num.

cubic lattice

Rotations: cubic box periodic BC in x,y,z

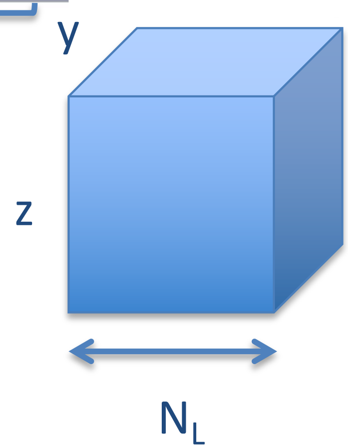
discrete Octahedral group O

24 elements

O	E	$8C_3$	$6C'_2$	$6C_4$	$3C_2=(C_4)^2$
A_1	+1	+1	+1	+1	+1
A_2	+1	+1	-1	-1	+1
E	+2	-1	0	0	+2
T_1	+3	0	-1	+1	-1
T_2	+3	0	+1	-1	-1

characters

Irreducible repr. : good quantum number
spin J: not good



Irreducible representation: representation of transformation where objects transform just within themselves (and can not be further reduced by block-diagonalization)

Relation between
J in continuum and irrep on the lattce

$O^{J,M}$: object that transforms under continuum R
according to spin J and projecion M,
for example interpolator

D = Wigner D matrix

$$RO^{J,M}R^{-1} = \sum_{M'} D^{J}_{MM'}(R^{-1})\ O^{J,M'}$$

Representation $O^{J,M}$

irreducible under continous rotational group SO(3)

REDUCIBLE under discrete Octahedral group O

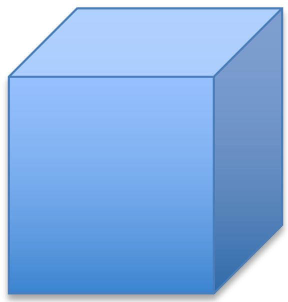
J	irrep Λ (dim)
0	A_1 (1)
1	T_1 (3)
2	E (2) + T_2 (3)
3	A_1 (1) + T_1 (3) + T_2 (3)

$O^{2,2}, O^{2,1}, O^{2,0}, O^{2,-1}, O^{2,-2}$

same thing

J	irrep
0	A_1
1, 3	T_1
2, 3	T_2
2	E
3	A_2

spins J>3 omitted

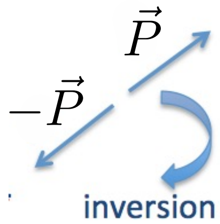


Conclusion:
given irrep contains different J

Symmetries are significantly reduced for $\vec{P} \neq 0$

continuum

Parity: NOT good



Rotations/reflections:

transformations that leave P invariant

rotations around P; little group U(1)

single-particle with momentum P and spin S:

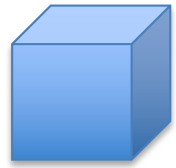
spin S of a particle: not good

only J=S+L : good

helicity : good $\lambda = \frac{\vec{S}\cdot\vec{P}}{|\vec{P}|}$

cubic lattice

Parity: NOT good



Rotations/reflections:

transformations that leave box and P invariant: Not much symmetry left !!

(or symmetries of box seen in moving frame)

P=(0,1,0): 8 elements

Irrep	I	$R(\pi)$	$R(3\pi/2)$	$R(\pi/2)$	Π	$R(\pi)\Pi$	$R(\pi/2)\Pi$	$R(3\pi/2)\Pi$

P=(1,1,0); 4 elements

Irrep	I	$R(\pi)$	Π	$R(\pi)\Pi$

irreps: good quantum numbers

helicity: not good

challenge: certain irrep gets contribution from both parities and several partial waves

Π is reflection in a plane that contains p; it preserves p

More in recent reviews

hadron spectrum from lattice:

N. Brambilla et al. 1907.07583, Phys. Rept

M. Mai, U. Meissner, C. Urbach, 2206.01477

N. Brambilla, 2111.10788

P. Bicudo, 2212.07793

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