The Advanced School on Mathematical Methods for Physicists

Problem Set 2: Feynman Integrals

IMSc, 2–6 July 2026

MCQ

Graph Theory, IBP & Feynman Integrals. Answer the following multiple-choice questions. Select the *single best answer* in each case.

1. *Kirchhoff's Theorem — Electrical Networks.* Kirchhoff's Matrix-Tree Theorem states that the number of spanning trees of a graph equals a cofactor of the graph Laplacian, and this directly counts the number of independent ways current can flow without loops.

   For K_4 (the complete graph on 4 nodes — imagine 4 nodes all pairwise wired with resistors), how many spanning trees are there?

   A) 4

   B) 8

   C) 16

   D) 24

2. A connected graph has loop number $L = 2$. One additional internal edge is inserted between two existing vertices without introducing a new vertex. The new graph can have at most

   (A) $L = 2$

   (B) $L = 3$

   (C) $L = 4$

   (D) $L = 5$

3. How many loops does this graph have?

   (A) 5

   (B) 6

   (C) 11

   (D) 16

4. A Feynman graph is *one-particle-irreducible* (1PI) if no single internal edge, when deleted, disconnects the graph – equivalently, no internal edge is a *bridge*. Suppose you "pinch" one propagator of a 1PI graph, i.e. you set its power to zero so that the corresponding edge is deleted from the graph (this is exactly what happens to a sub-topology generated in an IBP reduction). By how much does the loop number $L = E - V + 1$ necessarily decrease?

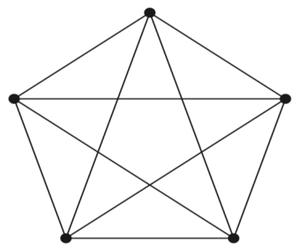

Figure 1: A graph whose loop number $L = E - V + C$ you are asked to determine, where E is the number of edges, V the number of vertices, and C the number of connected components.

(A) It cannot decrease; pinching a propagator never changes L.

(B) By exactly 1, always, because a 1PI graph has no bridges, so every edge lies on a cycle.

(C) By exactly 2, since each internal line carries two ends.

(D) It depends on the graph; some propagators pinch to $\Delta L = 1$ and others to $\Delta L = 2$.

5. The first Symanzik polynomial $\mathcal{U}(\alpha)$ of a connected graph admits the classic *spanning-tree* representation. Which of the following correctly states it?

   (A) $\mathcal{U} = \sum_T 1$, a sum of 1's over all spanning trees T of the graph.

   (B) $\mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e$, a sum over spanning trees T of the product of Feynman parameters of the edges *not* in T.

   (C) $\mathcal{U} = \sum_T \prod_{e \in T} \alpha_e$, a sum over spanning trees T of the product of Feynman parameters of the edges *in* T.

   (D) $\mathcal{U} = \prod_e \alpha_e \cdot$ (number of spanning trees).

6. The First Barnes Lemma of Problem 4 below expresses a Mellin–Barnes integral of four Gamma functions as a ratio of Gamma functions. Which classical hypergeometric identity is it secretly equivalent to?

   (A) The Euler integral representation of ${}_2F_1$.

   (B) Gauss's summation theorem for ${}_2F_1(a, b; c; 1)$.

   (C) The Pfaff transformation of ${}_2F_1$.

   (D) Kummer's relation for ${}_2F_1(a, b; 1 + a - b; -1)$.

Problem 1 *IBP Relations for the Two-Mass Bubble.* The family is

$$F(a_1,a_2) = \int \frac{d^d k}{D_1^{a_1}\, D_2^{a_2}}, \qquad D_1 = k^2 - m_1^2, \quad D_2 = (q-k)^2 - m_2^2. \tag{1}$$

The two IBP identities follow from $\int d^d k\, \partial_{k^\mu} \big(v^\mu / D_1^{a_1} D_2^{a_2} \big) = 0$ with $v^\mu \in \{k^\mu,\, q^\mu\}$. The relevant derivatives are

$$\partial_{k^\mu} D_1 = 2k^\mu, \qquad \partial_{k^\mu} D_2 = -2(q-k)^\mu. \tag{2}$$

Note that in this problem set we will work in Minkowski spacetime.

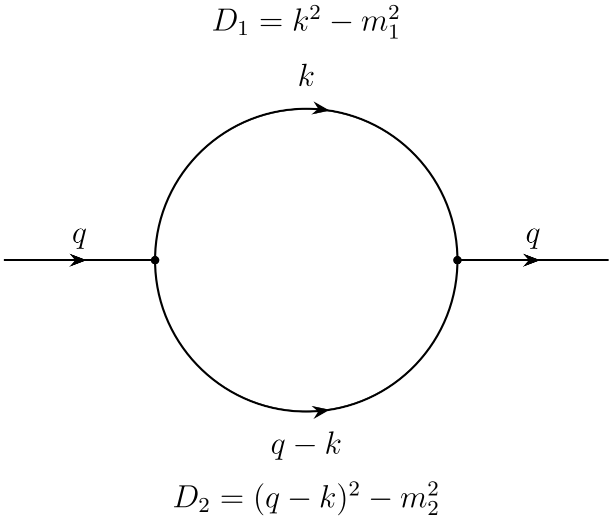

Figure 2: Two-Mass Bubble. Both propagators carry momentum flowing from left to right, ensuring $q = k + (q-k)$ at each vertex.

1. *IBP identities.* Using the above two IBP identities, show that $F(a_1,a_2)$ satisfies the following two identities

$$\begin{gathered} d - 2a_1 - a_2 - 2m_1^2\, a_1\, \mathbf{1}^+ - a_2\, \mathbf{2}^+ \big(\mathbf{1}^- - q^2 + m_1^2 + m_2^2 \big) = 0\,, \\ a_2 - a_1 - a_1\, \mathbf{1}^+ \big(q^2 + m_1^2 - m_2^2 - \mathbf{2}^- \big) - a_2\, \mathbf{2}^+ \big(\mathbf{1}^- - q^2 + m_1^2 - m_2^2 \big) = 0\,. \end{gathered} \tag{3}$$

   Here the standard notation for raising and lowering operators has been used,

$$\mathbf{1}^- \mathbf{2}^+\, F(a_1,a_2) = F(a_1 - 1, a_2 + 1).$$

2. *Master integrals.* Note that setting $a_2 = 0$ in $F(a_1,a_2)$ gives $F(a_1,0) = \int d^d k / (k^2 - m_1^2)^{a_1}$, a single-mass tadpole with no dependence on q. Likewise, setting $a_1 = 0$ gives $F(0,a_2)$; after the shift $k \to q - k$ this becomes $\int d^d k / (k^2 - m_2^2)^{a_2}$, again a single-mass tadpole independent of q. Using this observation together with the two IBP identities (3), show that every integral of the family, for integer a_1, a_2, can be written as a linear combination – with coefficients rational in d, q^2, m_1^2, m_2^2 – of three *master integrals*,

$$F_1 \equiv F(1,1), \qquad F_2 \equiv F(1,0), \qquad F_3 \equiv F(0,1).$$

   Explain why F_2 and F_3 (the tadpole sub-topologies) cannot be reduced any further, while F_1 (the genuine two-mass bubble) is the top-sector master. Work out explicitly the reduction of $F(2,1)$ onto $\{F_1, F_2, F_3\}$ as an example.

Problem 2 *Symanzik Polynomials from the Loop-Momentum Representation.* The parametric (Schwinger/Feynman-parameter) representation of a scalar L-loop integral with n_{int} propagators of powers ν_j, in D dimensions, reads

$$I \;=\; \frac{e^{L\varepsilon\gamma_E}}{\prod_{j=1}^{n_{\text{int}}}\Gamma(\nu_j)} \int_{\alpha_j\geq 0} d^{\,n_{\text{int}}}\alpha \left(\prod_{j=1}^{n_{\text{int}}}\alpha_j^{\nu_j-1}\right) \big[\mathcal{U}(\alpha)\big]^{-D/2} \exp\!\left(-\frac{\mathcal{F}(\alpha)}{\mathcal{U}(\alpha)}\right). \tag{4}$$

Here $\mathcal{U}$ is the *first* Symanzik polynomial, arising as the determinant of the quadratic form in loop momenta after Schwinger-parametrizing every propagator and integrating out $\ell_1,\ldots,\ell_L$ one at a time (a sequential Gaussian elimination); $\mathcal{F}$ is the *second* Symanzik polynomial, the residual momentum-independent piece left over after that elimination, rescaled by $\mathcal{U}$. The prefactor $e^{L\varepsilon\gamma_E}$ (with $D = 4-2\varepsilon$) is the customary normalization that removes Euler–Mascheroni constants order by order in the loop expansion. Equation (4) is the *pre-projective* form: the α_j are each integrated independently over the full half-line, before using the scaling (Cheng–Wu) identity to reduce to a projective integral with a $\delta\Big(1-\sum_j \alpha_j\Big)$ constraint.

How to use the `UF.m` package

1. `xx` — loop momenta

One symbol per independent loop momentum (treated as a formal scalar). `Length[xx]`$=$ L, the number of loops. E.g. one-loop: `{k}`; two-loop sunset: `{l1, l2}`.

2. `yy` — propagators, with a sign flip

Internally, `degree = -Sum[yy[i] x[i]]` must equal $\sum_i x_i D_i$, so

$$\boxed{\texttt{yy}_i = -D_i}, \qquad D_i = q_i^2 - m_i^2,$$

with $q_i =$ (loop momentum) + (external momenta) through line i.

Example (massless triangle, loop k, external p_1, p_2):

```
yy = {-k^2, -(k+p1)^2, -(k+p1+p2)^2}
```

Check: `Length[yy]` $= n_{\text{int}}$, the number of propagators.

3. `z` — kinematic substitutions

Expanding `yy` produces external-momentum dot products ($p_1{\cdot}p_2$, etc.); `z` rewrites these in terms of physical invariants, consistent with momentum conservation. *Loop*-momentum dot products need no rule — the elimination handles those automatically.

```
z = {p1^2 -> P1, p2^2 -> P2, p1 p2 -> (Q - P1 - P2)/2}
```

Leaving `z = {}` is fine too; $\mathcal{U}, \mathcal{F}$ just come out in terms of raw dot products.

Checklist

Check	Expected
`Length[xx]`	number of loops L
`Length[yy]`	number of propagators n_{int}
Sign of `yy`	each entry is $-D_i$
Output $\mathcal{U}$	homogeneous degree L
Output $\mathcal{F}$	homogeneous degree $L+1$

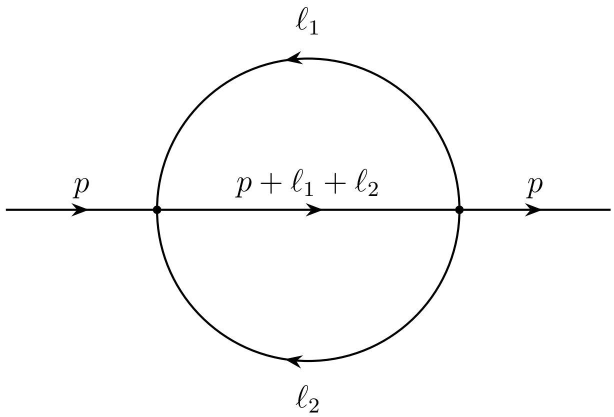

Figure 3: Two-loop sunset ("sunrise") diagram.

Consider the two-loop *sunset* ("sunrise") diagram: two independent loop momenta ℓ_1, ℓ_2, three internal massive propagators, and a single external momentum p flowing through the diagram,

$$D_1 = \ell_1^2 - m_1^2, \qquad D_2 = \ell_2^2 - m_2^2, \qquad D_3 = (\ell_1 + \ell_2 + p)^2 - m_3^2.$$

(a) *(Setting up the inputs.)* Using the routine

```
UF[xx_, yy_, z_]
```

from before, identify the correct inputs `xx` and `yy` for this diagram (recall $\texttt{yy}_i = -D_i$), and evaluate

```
UF[{l1, l2},
   {m1^2 - l1^2, m2^2 - l2^2, m3^2 - (l1 + l2 + p)^2},
   {p^2 -> P}] // Expand
```

(b) *(Homogeneity check.)* Confirm that the returned L equals 2, and check explicitly that $\mathcal{U}(\alpha)$ is homogeneous of degree $L = 2$ while $\mathcal{F}(\alpha)$ is homogeneous of degree $L + 1 = 3$ in the Feynman parameters x_1, x_2, x_3 – as required for the projective rescaling used in deriving (4).

(c) *(Sanity check.)* Set $m_1 = m_2 = m_3 = 0$ in your result for $\mathcal{F}$. What happens to the integral, and why is this the expected behavior for a massless sunset diagram with only one external scale p^2?

(d) Choose a Feynman graph of your choice and compute its first and second Symanzik polynomials, $\mathcal{U}$ and $\mathcal{F}$.

Problem 3 ***(Optional)*** *From Trees to Two-Forests: Why Gaussian Integration Stops at $k = 2$.* Recall from Problem 2 that, after Schwinger-parametrizing every propagator of an L-loop graph, the momentum-dependent part of the exponent is manifestly Gaussian in the loop momenta:

$$\sum_{i=1}^{n_{\rm int}} \alpha_i D_i \;=\; \ell^T A(\alpha)\,\ell \;+\; 2\,\ell^T B(\alpha)\,p \;+\; C(\alpha,p,m), \tag{5}$$

where $\ell = (\ell_1, \ldots, \ell_L)^T$ collects the independent loop momenta, p collects the independent external momenta, $A(\alpha)$ is a symmetric $L\times L$ matrix linear in the Feynman parameters α_i, $B(\alpha)$ is linear in α, and $C(\alpha,p,m)$ collects the mass terms $\sum_i \alpha_i m_i^2$ together with terms quadratic in the external momenta. Performing the d-dimensional Gaussian integral over $\ell_1, \ldots, \ell_L$ gives

$$\int d^{dL}\ell\; e^{-\left(\ell^T A\ell + 2\ell^T Bp + C\right)} \;\propto\; (\det A)^{-d/2}\,\exp\Big[(Bp)^T A^{-1}(Bp) - C\Big], \tag{6}$$

from which one reads off

$$\mathcal{U}(\alpha) = \det A(\alpha), \qquad \frac{\mathcal{F}(\alpha)}{\mathcal{U}(\alpha)} = C(\alpha,p,m) - (Bp)^T A^{-1}(Bp).$$

(a) Using the fact that $A(\alpha)$ is, up to a choice of loop basis, the weighted graph Laplacian reduced to the loop-momentum subspace, and quoting the weighted Matrix–Tree (Kirchhoff) theorem, explain why $\det A(\alpha)$ is necessarily a sum, with unit coefficients, over spanning trees of the graph – i.e. why $\mathcal{U}$ can never contain a term corresponding to a subgraph with two or more disconnected components.

(b) Using $A^{-1} = \mathrm{adj}(A)/\det A$, so that

$$(Bp)^T A^{-1}(Bp)\cdot\mathcal{U} \;=\; (Bp)^T\,\mathrm{adj}(A)\,(Bp),$$

argue that $\mathrm{adj}(A)$ – built from the $(L-1)\times(L-1)$ minors of A – computes a sum over spanning *two*-forests (pairs of trees partitioning all vertices into exactly two components), each weighted by the square of the total external momentum flowing between the two components. This is why $\mathcal{F}$ contains two-forest terms.

(c) *(The key point.)* Looking only at the structure of Eq. (6) – with no further graph theory – explain why no term corresponding to a spanning k-forest with $k > 2$ can ever appear in $\mathcal{U}$ or $\mathcal{F}$. *Hint:* what is the highest power of p (equivalently, of the source Bp) that can possibly appear on the right-hand side of Eq. (6)? Relate your answer to the vanishing of the third and higher cumulants of a Gaussian distribution.

(d) *(Explicit check.)* For the two-loop sunset diagram of Problem 2, work out $A(\alpha)$, $B(\alpha)$, $C(\alpha,p,m)$ explicitly from

$$\alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3, \qquad D_1 = \ell_1^2 - m_1^2,\;\; D_2 = \ell_2^2 - m_2^2,\;\; D_3 = (\ell_1+\ell_2+p)^2 - m_3^2,$$

and verify by direct computation of $\det A$ and $\mathrm{adj}(A)$ that you recover $\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and the single two-forest term $p^2\,\alpha_1\alpha_2\alpha_3$ found in $\mathcal{F}$ in Problem 2. Confirm explicitly that no term of degree $L+2 = 4$ (the signature of a three-forest) appears anywhere in your result.

(e) *(Physical interpretation.)* In one sentence, connect part (c) to the statement that a Gaussian (free) path integral only ever generates two-point Wick contractions. Why does this make it obvious, without any further computation, that the parametric representation of *any* Feynman graph – at any loop order, with any number of external legs – can only ever require one- and two-forests, and never more?

Problem 4 *First Barnes Lemma.* Consider the Mellin–Barnes contour integral

$$I(a,b,c,d) \;=\; \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c-z)\,\Gamma(d-z)\,dz,$$

where the contour separates the poles of $\Gamma(a+z)\Gamma(b+z)$ (at $z=-a-n,\,-b-n$, $n=0,1,2,\dots$) from those of $\Gamma(c-z)\Gamma(d-z)$ (at $z=c+n,\,d+n$).

(a) By closing the contour to the right and summing the residues at the poles of $\Gamma(c-z)\Gamma(d-z)$, show that

$$I(a,b,c,d) \;=\; \frac{\Gamma(a+c)\,\Gamma(b+c)\,\Gamma(a+d)\,\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

(b) Verify this result independently by noting that

$$I(a,b,c,d) \;=\; \mathcal{M}^{-1}_{z\to s}\Big[\,s^{z}\,\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c-z)\,\Gamma(d-z)\,\Big]\Big|_{s=1},$$

where $\mathcal{M}^{-1}_{z\to s}$ denotes the inverse Mellin transform in z, and confirm that the resulting expression is in fact independent of s, consistent with part (a).