Problem Set: Analyticity, Unitarity and Crossing

High Energy Physics / S-Matrix Theory

Problem 1: Gounaris–Sakurai Model and the Omnès Solution *Context: Study of resonances (ρ-meson) in $\pi\pi$ scattering using the Gounaris–Sakurai (GS) parameterization and the dispersion-theoretic Omnès solution.*

Take the Gounaris–Sakurai notebook. Follow all the steps outlined in the notebook and connect them directly to the course lecture notes.

(a) Show explicitly that the Omnès solution reproduces the electromagnetic pion Form Factor (FF) numerically.

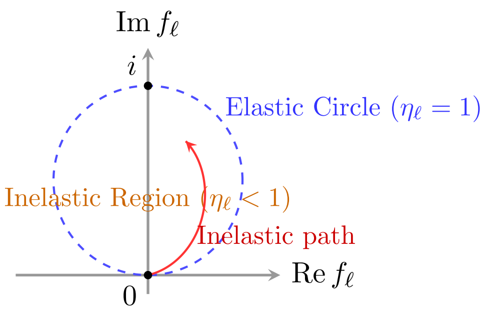
(b) Verify and check the consistency of equations (5.3.10) through (5.3.12) using the Gounaris–Sakurai parameterization.

Problem 2: Inelasticity and the Argand Plane for $f_0(s)$

The same notebook includes a specific model for describing inelasticity and the corresponding trajectory in the Argand plane for the partial-wave amplitude $f_0(s)$.

(a) Analyze the model, interpret the behavior of the amplitude, and explain your findings.

(b) Demonstrate and prove the relevant unitarity inequalities that govern the partial-wave amplitude when inelastic channels open up ($\eta_\ell < 1$).



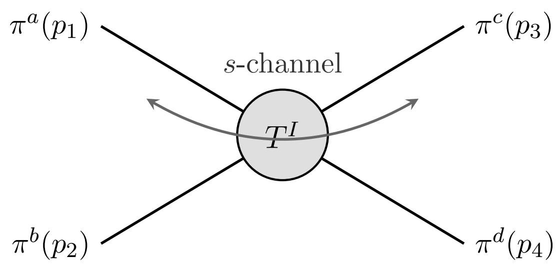
Problem 3: Kinematics for $\pi\pi$ Scattering

Verify the kinematic relations for elastic pion-pion ($\pi\pi$) scattering. Let s, t, u be the standard Mandelstam variables. Show that the s-channel scattering angle θ_s is given by:

$$\cos\theta_s = \frac{t-u}{s-4m_\pi^2}$$

and establish that the center-of-mass momentum squared q^2 satisfies:

$$q^2 = \frac{s-4m_\pi^2}{4}$$



Problem 4: Unequal Mass Case

Extend the kinematic and analytic analysis to the case of particles with unequal masses. Refer to and summarize the treatment presented in **Chapter 4, Section 3** of *Elementary Particle Physics* by A. D. Martin and T. D. Spearman. Note the appearance of kinematic singularities and constraints compared to the equal mass case.

Problem 5: Isospin Relations and Crossing Matrices

(a) Derive the isospin relations for $\pi\pi$ scattering using the standard formulation from Burkhardt involving $9j$ symbols, as well as the $6j$ symbols approach detailed by Neville.

(b) Establish and write down the explicit forms of the s-to-t, s-to-u, and t-to-u channel crossing matrices, denoted respectively as C_{st}, C_{su}, and C_{tu}.

Problem 6: Roy Representations and Unsubtracted Dispersion Relations

Start with the unsubtracted fixed-t dispersion relations for the $\pi\pi$ scattering amplitudes $T^I(s,t,u)$ of definite isospin I. By imposing crossing symmetry and analyticity, derive the **Roy representation**. Express the regular/"unknown" driving functions explicitly as integral contributions (sum-rules) over the absorptive parts of the amplitudes.

Problem 7: Identities for Crossing Matrices

Perform a rigorous algebraic check on the identities satisfied by the crossing matrices C_{st}, C_{su}, and C_{tu}. Verify properties such as involutivity ($C^2 = I$) and the consistency of cyclic transformations between the channels.

Problem 8: Roy Equation Kernels

Display the subtraction/driving functions $g_1(s,t)$, $g_2(s,t)$, and $g_3(s,t)$ that appear in the standard Roy equations. Explicitly evaluate and present some of the singular and regular algebraic kernels to highlight their crossing and analytic structure.

Problem 9: Asymptotic Behavior and the Froissart Bound

Discuss the asymptotic formula for the Legendre polynomials $P_\ell(x)$ in the non-physical domain where $x > 1$ (i.e., inside the Lehmann–Martin ellipse). Analyze this behavior specifically in the context of deriving and understanding the physical implications of the **Froissart bound** on total cross-sections.