The Advanced School on Mathematical Methods for Physicists

Problem Set 1: Feynman Integrals[1]

IMSc, 2–6 July, 2026

Problem 0 *Differential solid angle.* Using Gaussian integrals, show that the total solid angle in d dimensions is

$$\Omega_d = \int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}\,, \tag{1}$$

where $d = 2\omega$ throughout the rest of this problem set. *Hint*: evaluate $\int d^d x\, e^{-x^2}$ in both Cartesian and spherical coordinates, and compare the two answers.

1. *Consistency check.* Verify that our formula reproduces $\Omega_2 = 2\pi$ (the angle subtended by a circle) and $\Omega_3 = 4\pi$ (the surface area of a unit sphere).

2. *Numerical check — analytic continuation to non-integer d.* Our derivation only relies on the radial integral $\int_0^\infty dr\, r^{d-1} e^{-r^2} = \frac{1}{2}\Gamma(d/2)$, which is a perfectly good *convergent* integral for any real $d > 0$, not just integer d. Pick a non-integer value, say $d = 2.5$, and evaluate this radial integral numerically (e.g. with `scipy.integrate.quad` in Python, or `NIntegrate` in Mathematica). Compare your result against $\frac{1}{2}\Gamma(d/2)$, computed with a standard Gamma-function routine. Why does this check matter for the rest of the problem set, given that we will eventually set $d = 2\omega = 4 - \epsilon$ and take $\epsilon \to 0$?

3. *Exploratory (bonus).* Using your favourite plotting tool, plot Ω_d as a function of d for $d \in (0, 15)$. You should find that Ω_d is *not* monotonically increasing in d: it reaches a maximum around $d \approx 7.26$ and then decreases. Can you explain, at least qualitatively, why the "surface area of a unit sphere" should eventually shrink as the number of dimensions grows? *Hint*: think about how the volume of a unit ball is distributed between being "close to" and "far from" its centre as d increases.

Problem 1 *Integration in arbitrary dimensions.* Using Gaussian integration together with the Schwinger parametrisation [2] derive the following master formula, which will be used throughout the rest of this problem set:

$$\boxed{\int \frac{d^{2\omega} l}{(2\pi)^{2\omega}}\, \frac{1}{(l^2 + M^2 + 2\,l\cdot p)^A} = \frac{\Gamma(A-\omega)}{(4\pi)^\omega\, \Gamma(A)}\, \frac{1}{(M^2 - p^2)^{A-\omega}}\,.} \tag{2}$$

[1] *We will work in Euclidean space throughout this problem set.*

[2] Schwinger parametrisation/trick : $\frac{1}{D^a} = \frac{1}{\Gamma(a)} \int_0^\infty d\alpha\, \alpha^{a-1}\, e^{-\alpha D}$,

1. *Completing the square.* Complete the square in the denominator and shift the loop momentum l to eliminate the term linear in l. Show that this leaves an integral of the form $\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{(l^2+\Delta)^A}$, and identify Δ in terms of M^2 and p^2.

2. *Schwinger parametrisation.* Introduce a Schwinger parameter α for the shifted propagator $1/(l^2+\Delta)^A$, so that the l-integral becomes a purely Gaussian integral of $e^{-\alpha l^2}$ over 2ω dimensions.

3. *Gaussian integration.* Perform the Gaussian l-integral. *Hint*: using the solid-angle result from Problem 0, show that

$$\int d^{2\omega} l\, e^{-\alpha l^2} = \left(\frac{\pi}{\alpha}\right)^{\omega}, \tag{3}$$

and hence that $\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\, e^{-\alpha l^2} = \dfrac{1}{(4\pi)^{\omega}\alpha^{\omega}}$.

4. *Final integration.* You should now be left with a single integral over α of the form $\alpha^{A-\omega-1}e^{-\alpha\Delta}$. Perform this integral to recover the boxed formula above, and state the condition on A and ω required for the α-integral to converge.

5. *Numerical check.* Set $p=0$, so that the boxed formula collapses to

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{(l^2+M^2)^A} = \frac{\Gamma(A-\omega)}{(4\pi)^{\omega}\,\Gamma(A)} \frac{1}{(M^2)^{A-\omega}}\,. \tag{4}$$

   Pick convenient numerical values for which the left-hand side is manifestly convergent — e.g. $\omega = 1.5$ (so $d=3$), $A=2$, and $M^2=1$ — and evaluate the left-hand side directly as a d-dimensional radial integral:

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{(l^2+M^2)^A} = \frac{\Omega_d}{(2\pi)^d}\int_0^\infty dr\, \frac{r^{d-1}}{(r^2+M^2)^A}\,, \tag{5}$$

   using the Ω_d formula from Problem 0. Evaluate the radial integral numerically (`scipy.integrate.quad` or `NIntegrate`) and compare against the right-hand side, evaluated with a standard Gamma-function routine. Repeat for a couple of other choices of (ω, A, M^2) with $A>\omega$, to make sure the check is not a coincidence at a single point. *Note*: for $A\le\omega$ the radial integral diverges at large r, consistent with the pole structure of $\Gamma(A-\omega)$ on the right-hand side — this is your first hint of how UV divergences will show up as poles in ω (or, later, in ϵ) once we continue away from integer dimensions.

Problem 2 *Evaluation of the 'tadpole' diagram.* We now put the master formula to work by evaluating a low-order Feynman diagram in $\lambda\phi^4$ theory. Recall that dimensional regularisation requires us to evaluate Feynman integrals in 2ω dimensions, where the coupling constant λ is no longer dimensionless. It is convenient to trade λ for a dimensionless coupling λ_{new} via

$$\lambda_{\text{old}} = \lambda_{\text{new}}\,(\mu^2)^{2-\omega}\,, \tag{6}$$

where μ^2 is an arbitrary constant with dimensions of mass. Equivalently, we are evaluating Green's functions for the theory defined by the action

$$S_\omega = \int d^{2\omega}x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda_{\text{new}}}{4!}\,(\mu^2)^{2-\omega}\,\phi^4 \right]. \tag{7}$$

The Feynman rules for this theory match those in four dimensions, with three modifications: (1) scalar products between vectors are summed over all 2ω components; (2) loop integrals carry the measure $\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}$; and (3) the vertex factor $-\lambda$ is replaced by $-\lambda_{\text{new}}\,(\mu^2)^{2-\omega}$.

1. *Massless tadpole.* Show that

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\,\frac{1}{l^2} = 0\,. \tag{8}$$

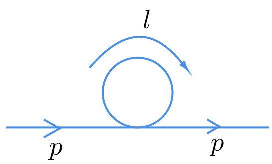
   How can this result hold given that the integral is both UV and IR divergent? Explain why the absence of any physical scale forces every scaleless integral to vanish in dimensional regularisation.

2. *Dimensional analysis.* Consider the integral

$$I(m) = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}\,\frac{1}{(l^2+m^2)^A}\,. \tag{9}$$

   Without evaluating the integral, use dimensional analysis to determine its dependence on m.

3. *Tadpole integrand.* Using the Feynman rules above, write down the one-loop integrand I_{tadpole} for the tadpole diagram shown below. *Hint*: don't forget the combinatorial symmetry factor associated with this diagram.



$\equiv\; I_{\text{tadpole}}$

4. *Evaluating the integral.* Apply the master dim-reg formula derived in Problem 1 (with $A = 1$ and no external momentum flowing through the loop) to show that

$$I_{\text{tadpole}} = -\frac{\lambda_{\text{new}}}{2}\,(\mu^2)^{2-\omega}\,\frac{\Gamma(1-\omega)}{(4\pi)^\omega}\,(m^2)^{\omega-1}\,. \tag{10}$$

5. *Pole structure.* Set $2\omega = 4-\epsilon$ and expand $\Gamma(1-\omega) = \Gamma(-1+\epsilon/2)$ about $\epsilon = 0$. *Hint*: use the recursion relation $\Gamma(z+1) = z\,\Gamma(z)$ to relate $\Gamma(-1+\epsilon/2)$ to $\Gamma(1+\epsilon/2)$, whose expansion you already know. Show that the divergent piece is

$$I_{\text{tadpole}}\Big|_{\text{div}} = \frac{\lambda_{\text{new}}\,m^2}{16\pi^2}\,\frac{1}{\epsilon} + \text{finite}\,. \tag{11}$$

6. *Interpretation.* Unlike the fish diagram of Problem 4, this divergence is proportional to m^2 rather than being a pure number. What kind of counterterm in the Lagrangian does this divergence require, and which physical parameter of the theory does it renormalise?

7. *Numerical check.* Use `Package-X`(or any package of you like) to evaluate I_{tadpole} in Mathematica, and check your result against parts 4 and 5.

Problem 3 *Feynman parameters.* Feynman parametrisation is a technique for evaluating the loop integrals that arise from Feynman diagrams with one or more loops. In this problem we derive the Feynman parametrisation formula for n propagators.

1. *Two-propagator identity.* One can check that
$$\frac{1}{AB} = \int_0^1 \frac{du}{\left(uA + (1-u)B\right)^2}\,.$$
Derive this identity from scratch using the Schwinger parametrisation. *Hint*: write $1/A$ and $1/B$ each as a Schwinger-parameter integral over α and β respectively, then change variables to an overall scale $s = \alpha + \beta$ and a fraction $u = \alpha/(\alpha+\beta)$; the s-integral should reduce to a standard Gamma-function integral.

2. *General n-propagator formula.* Generalise this construction to n propagators $D_1^{a_1} \cdots D_n^{a_n}$, each raised to an arbitrary power a_i, by introducing n independent Schwinger parameters $\alpha_1, \ldots, \alpha_n$. Show that
$$\frac{1}{D_1^{a_1} \cdots D_n^{a_n}} = \frac{\Gamma(a_1 + \cdots + a_n)}{\Gamma(a_1) \cdots \Gamma(a_n)} \int_0^1 dx_1 \cdots dx_n \, \frac{\delta(\sum_i x_i - 1) \prod_i x_i^{a_i - 1}}{(\sum_i x_i D_i)^{\sum_i a_i}}\,. \tag{12}$$

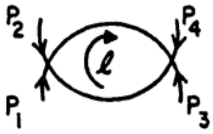
3. *Consistency check.* Verify that setting $n = 2$ and $a_1 = a_2 = 1$ in your general formula reproduces the identity from part 1.

4. *Simplex geometry (bonus).* The delta function $\delta(\sum_i x_i - 1)$ restricts the x_i to a simplex. For $n = 3$, what is the geometric shape of this region of integration? Plot it.

We will use the $n = 2$ case of this identity in the next problem, with $D_1 = l^2 + m^2$ and $D_2 = (l-p)^2 + m^2$ the two propagators of the 'fish' diagram.

Problem 4 *Fish diagram.*
The 'fish' (or 'bubble') diagram is the one-loop correction to the four-point vertex in $\lambda\phi^4$ theory: two propagators run between two vertices, with total external momentum p flowing through the loop.

1. *Fish-diagram integrand.* Write down the one-loop integrand for the fish diagram. *Hint*: this diagram has two vertex factors (from the Feynman rules of Problem 2) and a combinatorial symmetry factor of $1/2$ — make sure both appear in your integrand.



$$p_1 + p_2 + p_3 + p_4 = 0 \ .$$

Figure 1: Fish diagram

2. *Feynman parametrisation.* Using the Feynman parametrisation from the previous problem, together with the change of variable $l' = l - p(1-x)$, rewrite the integrand in the form
$$I_{\text{fish}} \propto \int dx \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{\left[l^2 + m^2 + p^2 x(1-x)\right]^2} \,. \tag{13}$$

3. *Loop integration.* Perform the loop integration using the master formula derived in Problem 1, and show that
$$I_{\text{fish}} = \frac{\lambda_{\text{new}}^2 \, (\mu^2)^{2(2-\omega)}}{2\,(4\pi)^{\omega}} \, \Gamma(2-\omega) \int_0^1 dx \left[m^2 + p^2 x(1-x)\right]^{\omega-2} . \tag{14}$$

4. *Pole structure.* Set $2\omega = 4 - \epsilon$ (i.e. $\omega = 2 - \epsilon/2$) and expand $\Gamma(2-\omega) = \Gamma(\epsilon/2)$ about $\epsilon = 0$. Show that the divergent piece takes the form
$$I_{\text{fish}}\Big|_{\text{div}} = \frac{\lambda_{\text{new}}^2}{16\pi^2} \frac{1}{\epsilon} + \text{finite} \,, \tag{15}$$
and write out the finite remainder explicitly (it will involve γ_E, $\ln 4\pi$, and an x-integral of $\ln\left[(m^2 + p^2 x(1-x))/\mu^2\right]$).

5. *Interpretation.* Notice that the residue of the $1/\epsilon$ pole is independent of both p^2 and m^2. Briefly explain why this had to be the case on physical grounds, and how this divergence is absorbed by a coupling-constant counterterm δ_λ in the renormalised Lagrangian.

6. *Four-point function (bonus).* The one-loop four-point function receives contributions from three channels: $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$. Can you write down the full one-loop four-point function in ϕ^4 theory as a sum of fish diagrams in the s-, t-, and u-channels?

7. *Minkowski continuation (bonus).* Can you analytically continue your result to Minkowski spacetime? *Hint*: after Wick rotation, the argument of the logarithm can become negative for $p^2 > 4m^2$ — what does this signal physically about the diagram?

8. *Numerical check.* Perform the loop integration in Mathematica (e.g. with `Package-X`) and check your result against parts 3 and 4.