

Hypergeometric function theory and Feynman Integrals at the interface

some recent Mathematica implementations

B. Ananthanarayan

Centre for High Energy Physics,
Indian Institute of Science, Bangalore



Overview

Introduction

Feynman integrals and hypergeometric functions

PART I : The `FeynGKZ.wl` package

PART II : The `Olsson.wl` package

PART III : The `AppellF2.wl` package

Future directions & bibliography

Gauss hypergeometric function ${}_2F_1$

► Pochhammer symbol

$$\begin{aligned}(x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)}, \\ &= x(x+1)\dots(x+n-1), \quad x \in \mathbb{C} \setminus \mathbb{Z}_0^-, n \in \mathbb{Z}_0^+ \\ (1)_n &= n!\end{aligned}$$

The Gauss ${}_2F_1(a, b; c; x)$:

$$\begin{aligned}{}_2F_1(a, b; c; x) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} x^n, \quad |x| < 1 \\ &= 1 + \frac{abx}{c} + \frac{a(a+1)b(b+1)x^2}{2c(c+1)} + O(x^3)\end{aligned}$$

- The $w = {}_2F_1$ hypergeometric function satisfies the ordinary differential equation (ODE)

$$x(1-x)\frac{d^2w}{dx^2} + [c - (a+b+1)x]\frac{dw}{dx} - abw = 0$$

- 3 singular points : 0, 1 and ∞ .

Method to find analytic continuations

The Gauss ${}_2F_1$

- Integral representation

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a} dt$$

- Analytic continuations of ${}_2F_1$: finding relations between solutions of the ODE around each singular point.
- Connecting the solutions around $x = 0$ and $x = 1$.
- Use the following relation

$$\begin{aligned} {}_2F_1(a, b; c; x) &= A {}_2F_1(a, b; a+b+1-c; 1-x) \\ &\quad + B(1-x)^{c-a-b} {}_2F_1(c-a, c-b; 1+c-a-b; 1-x). \end{aligned}$$

- Find A and B by substituting $x = 0$ and $x = 1$



$$\begin{aligned} {}_2F_1(a, b; c; x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b+1-c; 1-x) \\ &\quad + (1-x)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; 1+c-a-b; 1-x). \end{aligned}$$

Linear transformations

The Gauss ${}_2F_1$

- Similarly, analytic continuation around $x = \infty$:

$$\begin{aligned} {}_2F_1(a, b, c; x) &= (-x)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{x}\right) \\ &\quad + (-x)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{x}\right) \end{aligned}$$

- Pfaff-Euler transformations :

$${}_2F_1(a, b, c; x) = (1-x)^{-a} {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$$

$${}_2F_1(a, b, c; x) = (1-x)^{-b} {}_2F_1\left(b, c-a; c; \frac{x}{x-1}\right)$$

$${}_2F_1(a, b, c; x) = (1-x)^{c-a-b} {}_2F_1(c-a, c-b; c; x)$$

Definitions

- Appell F_2 and F_4

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}$$

valid for $|x| + |y| < 1$

$$F_4(a, b; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}$$

valid for $\sqrt{|x|} + \sqrt{|y|} < 1$

- Lauricella $F_C^{(3)}$

$$F_C^{(3)} = \sum_{n_1, n_2, n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3} (a_2)_{n_1+n_2+n_3}}{(c_1)_{n_1} (c_2)_{n_3} (c_3)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3!}$$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$

The momentum representation

- ▶ Typically involve tensor and colour structures in numerator - do tensor reduction, colour decomposition
- ▶ Calculate the scalar integrals
- ▶ Momentum representation:

$$I_{\Gamma}(\nu, D) = \int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^n (-q_j^2 + m_j^2)^{\nu_j}}$$

l : number of loops

D : the space-time dimension

$\nu = (\nu_1, \dots, \nu_n)$: propagator powers

k_r -s and q_j -s are the loop-momenta and internal-momenta for the Feynman graph Γ

q_j -s are combinations of external momentum and loop momentum.

Feynman graphs/diagrams

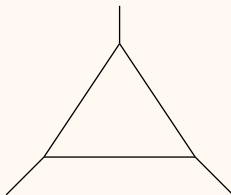
- ▶ Tadpole :



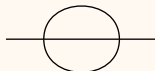
- ▶ bubble :



- ▶ 1-loop triangle :



- ▶ sunset :



detailed study of sunset integral can be found in [BA S. Friot, S. Ghosh '19 \[1\]](#)

Satisfies differential equations

- ▶ These Feynman integrals satisfy differential equation
- ▶ For example

$$\frac{d}{dk^2} \text{---} \bigcirc \text{---} + \frac{1}{2} \left[\frac{1}{k^2} - \frac{(D-3)}{k^2 + 4m^2} \right] \text{---} \bigcirc \text{---} + \frac{(D-2)}{4m^2} \left[\frac{1}{k^2} - \frac{1}{k^2 + 4m^2} \right] \text{---} \bigcirc = 0$$

- ▶ with proper boundary condition, the solution ($x = k^2/(4m^2)$)

$$\text{---} \bigcirc \text{---} = -\frac{(D-2)}{2m^2} \cdot \text{---} \bigcirc \cdot {}_2F_1 \left(1, 2 - \frac{D}{2}; \frac{3}{2}; -x \right)$$

- ▶ Tadpole can be expressed in terms of gamma functions

Relation to Feynman Integrals

- ▶ The dimension $d = 4 - 2\epsilon$
- ▶ One loop two-point function (B_0 function) : Anastasiou et. al. '00 [2]

$$F_4(1, \epsilon; 2 - \epsilon, \epsilon; x, y), \quad F_4(\epsilon, 2\epsilon - 1; \epsilon, \epsilon; x, y), \dots$$

with $x = m_1^2/p^2$, $y = m_2^2/p^2$

- ▶ One loop three-point function :

$$F_2(\epsilon + 1, 1, 1; \epsilon + 1, 2 - \epsilon; x, y), \\ F_2(1, 1 - \epsilon, 1; 1 - \epsilon, 2 - \epsilon; x, y), \dots$$

with $x = m_1^2/m_2^2$ and $y = q_1^2/m_2^2$

- ▶ The sunset integral with unequal masses : Berends et. al. '94 [3]

$$F_C^{(3)}(1, 2 - \epsilon; 2 - \epsilon, 2 - \epsilon, 2 - \epsilon; z_1, z_2, z_3), \\ F_C^{(3)}(1, \epsilon; 2 - \epsilon, \epsilon, 2 - \epsilon; z_1, z_2, z_3), \dots$$

with $z_1 = m_1^2/m_3^2$, $z_2 = m_2^2/m_3^2$ and $z_3 = p^2/m_3^2$

Domain of Convergences

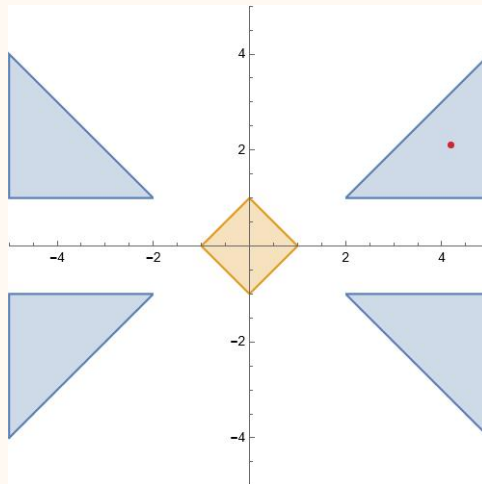


Figure: The defining domain of convergence of Appell F_2 (in orange), and of a analytic continuation of the same function that contains the red point (in blue) are plotted in real x - y plane

Mathematica packages

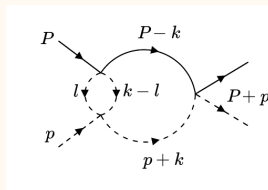
GKZ approach :: Mellin-Barnes method :: Method of Olsson :: multivariate hypergeometric functions

- ▶ [FeynGKZ.wl](#) : To express Feynman integrals in terms of multivariate hypergeometric functions [BA, S. Banik, S. Bera, S. Datta, '23](#) [4]
- ▶ [MBConicHulls.wl](#) : Study of N -fold Mellin-Barnes integrals [BA, S. Banik, S. Friot, S. Ghosh, '21](#) [5]
- ▶ [Olsson.wl](#) : Automated package to find ACs of MHFs [BA, S. Bera, S. Friot, T. Pathak, '21](#) [6]
- ▶ [AppellF2.wl](#), [AppellF1.wl](#), [AppellF3.wl](#) : Study of Appell F_2 [BA, S. Bera, S. Friot, O. Marichev, T. Pathak, '21](#) [7]
[S. Bera, T. Pathak, '24](#) [8]
- ▶ [LauricellaFD.wl](#), [LauricellaSaranFS.wl](#) : Numerical evaluation of triple variable Lauricella Saran $F_D^{(3)}$, $F_S^{(3)}$ functions [S. Bera, T. Pathak, '24](#) [8]
- ▶ [MultiHypExp.wl](#) : Series expansion of MHFs about their parameters [S. Bera, '22, '23](#) [9, 10]

Mathematica packages

Algebraic relation among Feynman integrals :: Method of regions :: Chisholm approximation

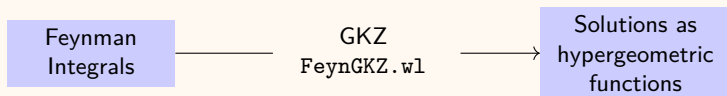
- ▶ **AlgRel.wl** : To find algebraic relations among Feynman integrals **BA, S. Bera, T. Pathak, '23** [11]
- ▶ **ASPIRE** : New approach to MoR **BA, A. Pal, S. Ramanan, R. Sarkar, '18** [12]
In the context of $\pi - K$ scattering at the threshold two loop fish diagram is considered



- ▶ **Chisholm D.wl** : To find rational approximant for bi-variate series **S. Bera, T. Pathak, '23** [13]

Part I

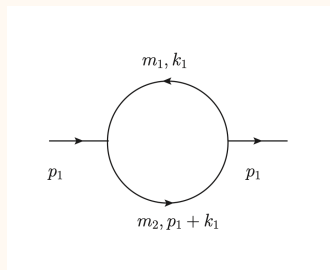
based on BA, S. Banik, S.Bera, S. Datta, '23 [4]



Work flow

- ▶ Feynman integral in Lee-Pomeransky (LP) representation can be thought of as solution of a set of partial differential equations
- ▶ These set of PDEs are known as Gel'fand-Kapranov-Zelevinsky (GKZ) systems
- ▶ Using the GKZ approach, hypergeometric series solution of these integrals can be obtained
- ▶ There are two different approaches
- ▶ **Algebraic** : GD method $\approx$ 'generalized Frobenius method'
- ▶ **Geometrically**: the triangulation method : the triangulation of the polytope associated with the LP polynomial is considered.
- ▶ **triangulation** (in 2D) : breaking a polygon into triangles.
Example: A rectangle could be broken in exactly two ways

Example



► $U = x_1 + x_2$

► $F = m_1^2 x_1^2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$

► $G = x_1 + x_2 + m_1^2 x_1^2 + (m_1^2 + s + m_2^2) x_1 x_2 + m_2^2 x_2^2$

►

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$$

Bubble diagram with two unequal masses (contd.)

Calculate the Γ -series:

```
In[7]:= SeriesSolution = SeriesRepresentation[Triangulations,2];
```

```
Prints => Unimodular Triangulation -> 2
```

```
Number of summation variables -> 2
```

```
Non-generic limit -> {z1 -> m1^2, z2 -> s + m1^2 + m2^2, z3 -> 1, z4 -> m2^2, z5 -> 1}
```

```
The series solution is the sum of following 3 terms.
```

```
Term 1 ::
```

$$\left(\left((-1)^{-n_1-n_2} \Gamma[-2+\epsilon+a_1-n_1-n_2] \Gamma[4-2\epsilon-a_1-a_2+n_2] \right. \right. \\ \left. \Gamma[a_2+2n_1+n_2] (m_1^2)^{2-\epsilon-a_1} \left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2} \right)^{n_1} \left(\frac{m_1^2}{s+m_1^2+m_2^2} \right)^{n_2} \right. \\ \left. (s+m_1^2+m_2^2)^{-a_2} \right) / (\Gamma[a_1] \Gamma[4-2\epsilon-a_1-a_2] \Gamma[a_2] \\ \Gamma[1+n_1] \Gamma[1+n_2]) \Bigg)$$

```
Term 2 ::
```

$$\left(\left((-1)^{-n_1-n_2} \Gamma[-2+\epsilon+a_2-n_1-n_2] \Gamma[4-2\epsilon-a_1-a_2+n_2] \right. \right. \\ \left. \Gamma[a_1+2n_1+n_2] (m_2^2)^{2-\epsilon-a_2} \left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2} \right)^{n_1} \left(\frac{m_2^2}{s+m_1^2+m_2^2} \right)^{n_2} \right. \\ \left. (s+m_1^2+m_2^2)^{-a_1} \right) / (\Gamma[a_1] \Gamma[4-2\epsilon-a_1-a_2] \Gamma[a_2] \\ \Gamma[1+n_1] \Gamma[1+n_2]) \Bigg)$$

```
Term 3 ::
```

$$\left(\left((-1)^{-n_1-n_2} \Gamma[2-\epsilon-a_2+n_1-n_2] \Gamma[2-\epsilon-a_1-n_1+n_2] \right. \right. \\ \left. \Gamma[-2+\epsilon+a_1+a_2+n_1+n_2] \left(\frac{m_1^2}{s+m_1^2+m_2^2} \right)^{n_1} \left(\frac{m_2^2}{s+m_1^2+m_2^2} \right)^{n_2} \right. \\ \left. (s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2} \right) / (\Gamma[a_1] \Gamma[4-2\epsilon-a_1-a_2] \\ \Gamma[a_2] \Gamma[1+n_1] \Gamma[1+n_2]) \Bigg)$$

```
Time Taken 0.066558 seconds
```

Bubble diagram with two unequal masses (contd.)

Check for an expression in terms of known hypergeometric functions using `Olsson.wl` :

```
In[8]:= GetClosedForm[SeriesSolution];
```

Prints ⇒ Closed form found with Olsson!

Term 1 ::

$$\frac{1}{\Gamma[a_1]} \Gamma[-2 + \epsilon + a_1] H_3\left[a_2, 4 - 2\epsilon - a_1 - a_2, 3 - \epsilon - a_1, \frac{m_1^2 m_2^2}{(s + m_1^2 + m_2^2)^2}, \frac{m_1^2}{s + m_1^2 + m_2^2}\right] m_1^4 (m_1^2)^{-\epsilon - a_1} (s + m_1^2 + m_2^2)^{-a_2}$$

Term 2 ::

$$\frac{1}{\Gamma[a_2]} \Gamma[-2 + \epsilon + a_2] H_3\left[a_1, 4 - 2\epsilon - a_1 - a_2, 3 - \epsilon - a_2, \frac{m_1^2 m_2^2}{(s + m_1^2 + m_2^2)^2}, \frac{m_2^2}{s + m_1^2 + m_2^2}\right] m_2^4 (m_2^2)^{-\epsilon - a_2} (s + m_1^2 + m_2^2)^{-a_1}$$

Term 3 ::

$$\left(\left(G1\left[-2 + \epsilon + a_1 + a_2, 2 - \epsilon - a_1, 2 - \epsilon - a_2, -\frac{m_2^2}{s + m_1^2 + m_2^2}, -\frac{m_1^2}{s + m_1^2 + m_2^2}\right] \Gamma[2 - \epsilon - a_1] \Gamma[2 - \epsilon - a_2] \right. \right. \\ \left. \Gamma[-2 + \epsilon + a_1 + a_2] (s + m_1^2 + m_2^2)^{2 - \epsilon - a_1 - a_2} \right) / (\Gamma[a_1] \Gamma[a_2])$$

Time Taken 0.05827 seconds

Summary

- ▶ Feynman integrals are solutions of GKZ hypergeometric system
- ▶ Feynman integrals can be expressed in terms of multivariate hypergeometric functions (MHFs)
- ▶ Two equivalent approaches : Gröbner deformation method and triangulation approach
- ▶ We have also studied their interconnection in [4]
- ▶ The power of propagators and the dimensional parameter appear as Pochhammer parameters
- ▶ The ratio of scales appear as variable of MHFs
- ▶ One then goes on to find ACs or series expansion of MHFs about the dimensional parameter

Part II

based on BA, S. Friot, S. Bera, T. Pathak, '21 [6]

The `Olsson.wl` package

Olsson - ROC2

(B. Ananthanarayan, S. Friot, S. Bera, T. Pathak) [6]

- `ROC2.wl` : an independent package that finds the region of convergence (ROC) of a double hypergeometric series, is a part of `Olsson.wl`

- The command `Olsson` takes the arguments as

```
In[1]:= Olsson[q,summation_index_List,expression,options]
```

- `summation_index_List` is the list of summation indices and `q` is an integer that can take value from 1 to `Length[summation_index_List]`
- The available options of `Olsson.wl` are

```
sum,one,inf,PET1,PET2,PET3,sim,roc
```

Commands and options of `Olsson.wl`

`sum, one, inf, PET1, PET2, PET3, sim, roc`

- ▶ The option `sum` takes the summation of the `expression` wrt `q`-th entry of the `summation_index_List`
- ▶ The option `one` performs the AC of ${}_2F_1(\dots, z)$ around $z = 1$

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1; 1-z) \end{aligned}$$

- ▶ The option `inf` performs the AC of ${}_2F_1(\dots, z)$ around $z = \infty$

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{z}\right) \end{aligned}$$

and similar AC from ${}_pF_{p-1}(\dots, z)$

Commands and options of `Olsson.wl`

- ▶ The options `PET1`, `PET2`, `PET3` does the Pfaff-Euler transformations

$${}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right)$$

$${}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1\left(b, c - a; c; \frac{z}{z - 1}\right)$$

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z)$$

- ▶ `sim` option simplifies the gamma functions, Pochhammer symbols assuming that each of the summation index belongs to $\mathbb{N}_0$
- ▶ The `roc` option find the region of convergence (ROC) of the final expression, provided they are double hypergeometric functions
- ▶ This option calls the `ROC2.wl` package to find the ROC

Demonstration of Olsson.wl

Options `sum`, `inf`

- ▶ Storing the summand in the variable `F2`

$$\text{In}[2]:= F2 = \frac{\text{Pochhammer}[a, m+n] \text{Pochhammer}[b1, m] \text{Pochhammer}[b2, n]}{\text{Pochhammer}[c1, m] \text{Pochhammer}[c2, n]} \frac{x^m y^n}{m! n!};$$

- ▶ The `sum` can be used as

`In[3]:= Olsson[1, {m, n}, F2, sum → True]`

$$\text{Out}[3]= \frac{y^n \text{HypergeometricPFQ}[\{b1, a+n\}, \{c1\}, x] \text{Pochhammer}[a, n] \text{Pochhammer}[b2, n]}{n! \text{Pochhammer}[c2, n]}$$

- ▶ The option `inf` can be used as

`In[4]:= Olsson[1, {m, n}, F2, inf → True]`

$$\begin{aligned} \text{Out}[4]= & ((-x)^{-b1} y^n \text{Gamma}[c1] \text{Gamma}[a-b1+n] \text{HypergeometricPFQ}[\dots, \frac{1}{x}] \dots) / (n! \\ & \text{Gamma}[-b1+c1] \text{Gamma}[a+n] \text{Pochhammer}[c2, n]) \\ & + ((-x)^{-a-n} y^n \text{Gamma}[c1] \text{Gamma}[-a+b1-n] \text{HypergeometricPFQ}[\dots, \frac{1}{x}] \dots) / (n! \\ & \text{Gamma}[b1] \text{Gamma}[-a+c1-n] \text{Pochhammer}[c2, n]) \end{aligned}$$

Demonstration of Olsson.wl

Options `sim`, `roc`

- The output can simplified using `sim` option

```
In[5]:= Olsson[1,{m,n}, F2 ,inf→True, sim→True]
```

```
Out[5]=
```

$$\frac{(-x)^{-a-n} x^{-m} y^n \Gamma[a+b1] \Gamma[c1] \text{Pochhammer}[a, m+n] \text{Pochhammer}[b2, n] \text{Pochhammer}[1+a-c1, m+n]}{m! n! \Gamma[b1] \Gamma[-a+c1] \text{Pochhammer}[1+a-b1, m+n] \text{Pochhammer}[c2, n]} \\ + \frac{(-1)^n (-x)^{-b1} x^{-m} y^n \Gamma[a-b1] \Gamma[c1] \text{Pochhammer}[b1, m] \text{Pochhammer}[b2, n] \text{Pochhammer}[1+b1-c1, m]}{m! n! \Gamma[a] \Gamma[-b1+c1] \text{Pochhammer}[1-a+b1, m-n] \text{Pochhammer}[c2, n]}$$

- We have obtained the AC of F_2 around $(\infty, 0)$

- The associated ROC can be found using `roc` option

```
In[6]:= Olsson[1,{m,n}, F2 ,inf→True, sim→True, roc→True]
```

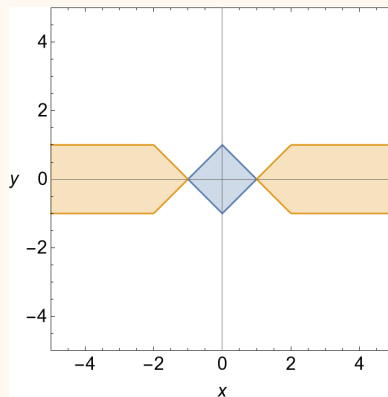
```
Out[6]=
```

$$\left\{ \frac{1}{\text{Abs}[x]} < 1 \&\& \text{Abs}\left[\frac{y}{x}\right] < 1 \&\& \text{Abs}\left[\frac{y}{x}\right] < 1 - \frac{1}{\text{Abs}[x]} \right. \\ \left. \&\& \frac{1}{\text{Abs}[x]} < 1 \&\& \text{Abs}[y] < 1 \&\& \text{Abs}[y] < -1 + \text{Abs}[x], \dots \right\}$$

Demonstration of Olsson.wl

ROC

- The ROCs plotted in real $x - y$ plane :



- The other options [one](#), [PET1](#), [PET2](#), [PET3](#) work similarly.
- repetitive use of these options can be made to find new ACs.

- ▶ the resulting series can be recognized using the `serrecog` or `serrecog2var` command

```
In[7]:= Plus@@(serrecog2var[{m,n},#]&/@(List@@Last[%6]))
```

```
Out[7]=
```

$$\frac{(-x)^{-b1} \Gamma[a-b1] \Gamma[c1]}{\Gamma[a] \Gamma[-b1+c1]} \text{FTilde}[\{\dots\}, \{\frac{1}{x}, -y\}]$$

$$+ \frac{(-x)^{-a} \Gamma[-a+b1] \Gamma[c1]}{\Gamma[b1] \Gamma[-a+c1]} \text{KdF}[\{\dots\}, \{\frac{1}{x}, -\frac{y}{x}\}]$$

- ▶ we recover the well-known analytic continuation of Appell F_2 .
- ▶ The `serrecog2var` command can recognize all 14 Appell-Horn series in two variables.
- ▶ The `serrecog` command can recognize bi-variate KdF, mirror-KdF (FTilde), Lauricella functions in any number of variables.

Part III

based on **BA**, S. Bera, S. Friot, O. Marichev and T. Pathak, '21 [7]

The AppellF2.wl package

Usage and demonstration

AppellF2

(B. Ananthanarayan, S. Bera, S. Friot, O. Marichev and T. Pathak [7])

- It can find the value of F_2 for **generic complex values of Pochhammer parameters** and arbitrary **real values of x, y except the points on the singular lines**

- Usage :

```
In[8]:= AppellF2[a,b1,b2,c1,c2,x,y,precision,terms,F2show→ True]
```

- For example,

```
In[9]:= AppellF2[2.2345,3.363,0.242,8.3452,0.657,-2.311,5.322,
10,100,F2show→ True]
```

```
Out[9]=
```

```
0.09333639793-0.06847416686 I
```

- Other commands

`F2findall, F2expose, F2R0C, F2evaluate`

Challenges in numerical evaluation

- ▶ We found a total of 44 ACs for Appell F_2
- ▶ All the ACs should obey the cut structures of F_2
- ▶ The cut of F_2 lies from 1 to ∞ along the real axis for each of the variables

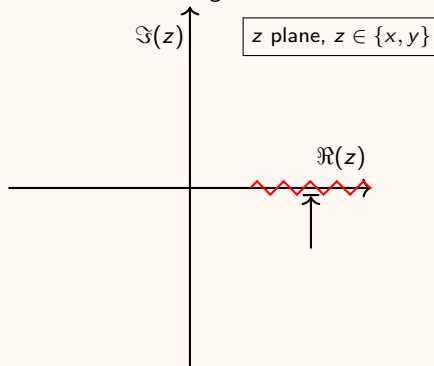


Figure: The red wiggly line denotes the cut of Appell F_2 and the arrow indicates the path of approach when the function is evaluated on the cut

- ▶ The value of F_2 on the cut is evaluated with ' $-i\epsilon$ prescription'

For x-cut,
$$F_2[\dots, x, y] = \lim_{\epsilon \rightarrow 0^+} F_2[\dots, x - i\epsilon, y]$$

Future directions

- ▶ Dispersion relation and Feynman integrals
- ▶ Hodge structure of Feynman integrals
- ▶ Relation with algebraic geometry, number theory, combinatorics
- ▶ Iterated Chen Integrals
- ▶ Theory of differential equations
- ▶ Theory of chords
- ▶ ...

References I

- [1] B. Ananthanarayan, Samuel Friot, and Shayan Ghosh. *Eur. Phys. J. C* 80.7 (2020), p. 606. arXiv: 1911.10096 [hep-ph].
- [2] C. Anastasiou, E. W. Nigel Glover, and C. Oleari. *Nucl. Phys. B* 572 (2000), pp. 307–360. arXiv: hep-ph/9907494.
- [3] Frits A. Berends et al. *Z. Phys. C* 63 (1994), pp. 227–234.
- [4] B. Ananthanarayan et al. *Comput. Phys. Commun.* 287 (2023), p. 108699. arXiv: 2211.01285 [hep-th].
- [5] B. Ananthanarayan et al. *Phys. Rev. Lett.* 127.15 (2021), p. 151601. arXiv: 2012.15108 [hep-th].
- [6] B. Ananthanarayan et al. *accepted for publication in CPC* (Dec. 2021). arXiv: 2201.01189 [cs.MS].
- [7] B. Ananthanarayan et al. *Comput. Phys. Commun.* 284 (2023), p. 108589. arXiv: 2111.05798 [math.CA].
- [8] Souvik Bera and Tanay Pathak. (Mar. 2024). arXiv: 2403.02237 [hep-ph].
- [9] Souvik Bera. *Nucl. Phys. B* 989 (2023), p. 116145. arXiv: 2208.01000 [math-ph].

References II

- [10] Souvik Bera. *Comput. Phys. Commun.* 297 (2024), p. 109060. arXiv: 2306.11718 [hep-th].
- [11] B. Ananthanarayan, Souvik Bera, and Tanay Pathak. *Nucl. Phys. B* 995 (2023), p. 116345. arXiv: 2307.04852 [hep-ph].
- [12] B. Ananthanarayan et al. *Eur. Phys. J. C* 79.1 (2019), p. 57. arXiv: 1810.06270 [hep-ph].
- [13] Souvik Bera and Tanay Pathak. (Sept. 2023). arXiv: 2309.07687 [cs.MS].
- [14] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner deformations of hypergeometric differential equations*. Vol. 6. Springer Science & Business Media, 2013.
- [15] Leonardo de la Cruz. *JHEP* 12 (2019), p. 123. arXiv: 1907.00507 [math-ph].
- [16] René Pascal Klausen. *JHEP* 04 (2020), p. 121. arXiv: 1910.08651 [hep-th].

References III

- [17] Jörg Rambau. “TOPCOM: Triangulations of Point Configurations and Oriented Matroids”. *Proceedings of the International Congress of Mathematical Software*. 2002. URL: <http://www.zib.de/PaperWeb/abstracts/ZR-02-17>.

Back up slides

Feynman Integral

- Momentum representation:

$$I_{\Gamma}(\nu, D) = \int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^n (-q_j^2 + m_j^2)^{\nu_j}}$$

l : number of loops; D : the space-time dimension; $\nu = (\nu_1, \dots, \nu_n)$: propagator powers

k_r -s and q_j -s are the loop-momenta and internal-momenta for the Feynman graph Γ .

- Lee-Pomeransky representation:

$$I_{\Gamma}(\nu, D) = \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{(l+1)D}{2} - \sum_i \nu_i)} \left(\prod_{i=1}^n \int_{\alpha_i=0}^{\infty} \frac{d\alpha_i \alpha_i^{\nu_i-1}}{\Gamma(\nu_i)} \right) G(\alpha)^{-\frac{D}{2}}$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

The Lee-Pomeransky representation (contd.)

- Generalized Feynman integral:

$$I_{G_z}(\nu, \nu_0) = \int_{\mathbb{R}_+^n} d\alpha \alpha^{\nu-1} G_z(\alpha)^{-\nu_0}$$

where, $\nu_0 = \frac{D}{2}$

- Generalized G -polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j}$$

$z_j \rightarrow$ generic/indeterminate

- Construct

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix}$$

In layman's terms

- ▶ Start with Feynman integral

$$I_{G_z}(\nu, \nu_0)$$

- ▶ Find its associated PDEs : $H_{\mathcal{A}}(\nu, \nu_0)$

$$\Rightarrow H_{\mathcal{A}}(\nu, \nu_0) I_{G_z}(\nu, \nu_0) = 0$$

- ▶ $H_{\mathcal{A}}(\nu, \nu_0)$ is called Gel'fand-Kapranov-Zelevinsky (GKZ) system or $\mathcal{A}$ -hypergeometric system
- ▶ Solve the PDEs:
 - ▶ Algebraic way : Gröbner deformation method (GD) (Saito, Sturmfels and Takayama [14], de la Cruz [15])
 - ▶ Geometric way : Triangulation method (Klausen [16])
- ▶ Both are equivalent
- ▶ GD $\approx$ 'generalized Frobenius method'

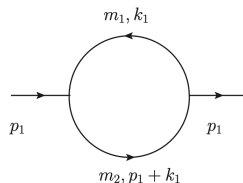
$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[u+v]_{u_+}} z^{u+v}$$

Bubble diagram with two unequal masses (contd.)

FeynGKZ

(B. Ananthanarayan, S. Banik, S. Bera, S. Datta [4])

Load the package
and its dependencies



```
In[3] := MomentumRep = {{k1, m1, a1}, {p1 + k1, m2, a2}};  
LoopMomenta = {k1};  
InvariantList = {p1^2 -> -s};  
Dim = 4 - 2ε;  
Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$ -matrix:

```
In[4] := FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta,
InvariantList, Dim, Prefactor}, UseMB → False];
```

Prints ⇒ The Symanzik polynomials $\rightarrow U = x_1 + x_2$
 $, F = m_1^2 x_1^2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$

The Lee-Pomeransky polynomial $\rightarrow G =$
 $x_1 + m_1^2 x_1^2 + x_2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$

The associated $\mathcal{A}$ -matrix $\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has $\text{codim} = 2$.

Normalized Volume of the associated Newton Polytope $\rightarrow 3$

Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations [J. Rambau \[17\]](#)

```
In[5] := Triangulations = FindTriangulations[FindAMatrixOut];
```

```
Prints => Finding all regular triangulations ...
Found 5 Regular Triangulations, out of which 3 are Unimodular
The 3 Unimodular Regular Triangulations ->
1 :: {{1,2,3},{2,3,4},{3,4,5}}
2 :: {{1,2,3},{2,4,5},{2,3,5}}
3 :: {{2,4,5},{1,3,5},{1,2,5}}
Time Taken 0.126965 seconds
```

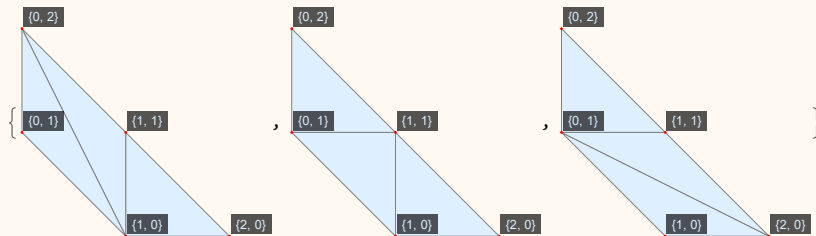


Figure: Visualization of 3 unimodular regular triangulations

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ -series terms numerically:

```
In[9] := SumLim = 30;  
ParameterSub = { $\epsilon \rightarrow 0.001$ ,  $a_1 \rightarrow 1$ ,  $a_2 \rightarrow 1$ ,  $s \rightarrow 10$ ,  $m_1 \rightarrow 0.4$ ,  $m_2 \rightarrow 0.3$ };  
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

```
Prints  $\Rightarrow$  Numerical result = 997.382  
Time Taken 0.222572 seconds
```